ON GENERATING FUNCTIONS

O. R. AINSWORTH
University of Alabama, University, Alabama 35486

Theorem. Consider the following three statements:

$$\psi(x,t) = \sum_{n=0}^{\infty} \phi_n(x)t^n , \tag{1}$$

$$\ln \psi(x,t) = \sum_{n=1}^{\infty} \frac{A_n(x)t^n}{n} , \tag{2}$$

$$n\phi_n(x) = \sum_{k=1}^{n} A_k(x)\phi_{n-k}(x) . \tag{3}$$

Any two of these statements imply the third.

Proof. For convenience in sum manipulation, let us define $A_0 = 1$ so that (3) becomes

$$(n+1)\phi_n(x) = \sum_{k=0}^{n} A_k(x)\phi_{n-k}(x). \tag{4}$$

We also normalize the $\phi_n(x)$ so that $\phi_0(x) = 1$.

Now assume that (1) and (4) are true; then from (4) we have

$$\sum_{n=0}^{\infty} (n+1)\phi_n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_k \phi_{n-k} t^n ,$$

or

$$\frac{d}{dt}\left[t \sum_{n=0}^{\infty} \phi_n t^n\right] = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} A_k \phi_n t^{n+k} .$$

Hence by (1)

$$\frac{d}{dt}\left[t\psi(x,t)\right] = \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^k ,$$

or

$$\frac{d}{dt}\left[t\psi(x,t)\right] = \psi(x,t) \sum_{k=0}^{\infty} A_k t^k .$$

Therefore

$$\frac{\frac{d}{dt}[t\psi(x,t)]}{t\psi(x,t)} = \sum_{k=0}^{\infty} A_k t^{k-1} ,$$

or, by integration,

$$\ln [t\psi(x,t)] = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} + \ln t + K(x).$$

Hence

$$\ln \psi(x,t) = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} + K(x)\,.$$

We may assume $K(x) = 0$ since we assume the $\phi_k(x)$ do not all have a common factor.

$$\therefore \ln \psi(x,t) = \sum_{k=1}^{\infty} \frac{A_k t^k}{k}\,,$$

which is statement (2).

If we assume (2) and (4) are true, then we have from (4)

$$\sum_{n=0}^{\infty} (n+1)\phi_n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_k \phi_{n-k} t^n = \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^k\,,$$

or

$$\frac{d}{dt}\left[t \sum_{n=0}^{\infty} \phi_n t^n\right] = t \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^{k-1}\,.$$

Divide and integrate, and we obtain

$$\ln\left[t \sum_{n=0} \phi_n t^n\right] = \sum_{k=1} \frac{A_k t^k}{k} + \ln t + \ln K(x).$$

Therefore, using (2),

$$\sum_{n=0}^{\infty} \phi_n(x) t^n = K(x)\psi(x,t)\,. \tag{5}$$

From (2), $\ln \psi(x,0) = 0$, so that $\psi(x,0) = 1$. Let $t \to 0$ in (5) and we get $\phi_0(x) = K(x)$, so $K(x) = 1$ since $\phi_0(x) = 1$. Hence

$$\sum_{n=0}^{\infty} \phi_n t^n = \psi(x,t)\,,$$

which is statement (1).

If we assume (1) and (2) are true, we get

$$\ln\,[t\psi(x,t)] = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} + A_0 \ln t$$

by adding $\ln t$ to both sides of (2) and remembering that $A_0 = 1$. Replacing $\psi(x,t)$ by its sum given in (1) and differentiating with respect to t,

$$\frac{d}{dt} \sum_{n=0}^{\infty} \phi_n t^{n+1} = \sum_{n=0}^{\infty} \phi_n t^{n+1} \sum_{k=1}^{\infty} A_k t^{k-1} + \frac{A_0}{t}\,.$$

$$\sum_{n=0}^{\infty} (n+1)\phi_n t^n = \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \phi_{n-k} A_k t^n\,.$$

Equating coefficients of t^n,

$$(n+1)\phi_n = \sum_{k=0}^{n} \phi_{n-k} A_k\,,$$

which is (4).

By rewording the previous theorem, we obtain this rendition:
If $\psi = \Sigma\phi_n t^n$, so that $t\psi = \Sigma\phi_n t^{n+1}$, then

$$e^{t\psi} = \sum \Theta_n t^n, \quad \text{where} \quad n\Theta_n = \sum_{k=1}^{n} k\phi_{k-1}\phi_{n-k}.$$

This naturally leads to all manner of strange generating functions. Omitting the trivial intervening steps, we list a small sample and note it is mildly surprising that the left-hand side should generate such a nice set of coefficients.

1| $$\exp\{t\}\exp\{xt\}\exp\{J_0(t\sqrt{1-x^2})\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n} \left(\frac{k}{(k-1)!}\right) P_{k-1}\phi_{n-k}.$$

2| $$\exp\left\{t(1-2xt+t^2)^{-1/2}\right\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n} kP_{k-1}\phi_{n-k}.$$

3| $$\exp\{t(1-t)^{1-\alpha-\beta}\}\exp\left\{{}_2F_1\left[\frac{1+\alpha+\beta}{2}, 1+\frac{\alpha+\beta}{2}; 1+\alpha; \frac{2t(x-1)}{(1-t)^2}\right]\right\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n} \frac{k(\alpha+\beta+1)_{k-1}}{(1+\alpha)_{k-1}} P_{k-1}^{\alpha,\beta}\phi_{n-k}.$$

4| $$\exp\{t^2(e^t-1)^{-1}\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n} \left(\frac{k}{(k-1)!}\right) B_{k-1}\phi_{n-k}.$$

In these equations, P_n and $P_n^{\alpha,\beta}$ are the Legendre and Jacobi polynomials, respectively, and B_n are the Bernoulli numbers. The ϕ_n are polynomials of degree n except in 4|.

The class of integrals easily obtained from these generating functions should delight any collector of the esoteric.

We close with two direct applications of the Theorem. Both are known, but the derivation is quite simplified.

5| Since $$(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha} t^n,$$

and $$-(1+\alpha)\ln(1-t) - \frac{xt}{1-t} = \sum_{n=0}^{\infty}\left[\frac{1+\alpha-x(n+1)}{n+1}\right]t^{n+1},$$

then $$nL_n = \sum_{k=1}^{n} (1+\alpha-kx)L_{n-k}^{\alpha},$$

where L_n^{α} are the Laguerre polynomials.

6| Since $$(1-2t\cos x+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos x)t^n \quad \text{and} \quad -\tfrac{1}{2}\ln(1-2t\cos x+t^2) = \sum_{r=1}^{\infty}\frac{t^r\cos rx}{r},$$

then $$(n+1)P_n(\cos x) = \sum_{k=0}^{n} \cos kx P_{n-k}(\cos x).$$

This work was supported in part by a grant from the Research Grant Committee of the University of Alabama, Project 763.

****