LIMITS OF QUOTIENTS FOR THE CONVOLVED FIBONACCI SEQUENCE AND RELATED SEQUENCES

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If $\{F_n\}_{n=1}^{\infty}$ is the sequence of Fibonacci numbers defined recursively by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 3$$

then $C_1(x)$, the generating function for the sequence $\{F_n\}_{n=1}^{\infty}$, is given by

(1) $$C_1(x) = (1-x-x^2)^{-1} = \sum_{i=0}^{\infty} F_{i+1}x^i .$$

Letting $C_n(x)$ be the generating function for the Cauchy convolution product of $C_1(x)$ with itself n times and $F_{i+1}^{(n)}$ be the coefficient of x^i in the n^{th} convolution, we have

(2) $$C_n^{(x)} = (1-x-x^2)^{-n} = \sum_{i=0}^{\infty} F_{i+1}^{(n)}x^i, \quad n \geqslant 1.$$

In a personal communique, V.E. Hoggatt, Jr., pointed out that he and Marjorie Bicknell have shown that

(3) $$\lim_{n\to\infty} \frac{F_{n+1}^{(r)}}{F_n^{(r)}} = \alpha$$

and

(4) $$\lim_{n\to\infty} \frac{F_n^{(r)}}{F_n^{(r+1)}} = 0,$$

where $\alpha = (1+\sqrt{5})/2$.

An immediate consequence of (3) is

(5) $$\lim_{n\to\infty} \frac{F_{n+k}^{(r)}}{F_{n+m}^{(r)}} = \alpha^{k-m}$$

while by using (4), we obtain

(6) $$\lim_{n\to\infty} \frac{F_{n+k}^{(r)}}{F_{n+m}^{(r+1)}} = 0.$$

The purpose of this note is to extend the results of (3) and (4) to the columns of the convolution array formed by a sequence of generalized Fibonacci numbers as well as to the array generated by the numerator polynomials of the generating functions for the row sequences associated with the convolution array formed by the given sequence of generalized Fibonacci numbers.

The sequence $\{H_n\}_{n=1}^{\infty}$ of generalized Fibonacci numbers defined recursively by

$$H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geqslant 3$$

has generating function $C_1^*(x)$ given by

$$C_1^*(x) = \sum_{i=0}^{\infty} H_{i+1}x^i = \frac{1+(P-1)x}{1-x-x^2} = \sum_{i=0}^{\infty} (F_{i+1}+(P-1)F_i)x^i . \tag{7}$$

Using $C_n^*(x)$ for the Cauchy convolution product of $C_1^*(x)$ with itself n times and $H_{i+1}^{(n)}$ for the coefficient of x^i in the n^{th} convolution, we have

$$C_n^*(x) = \sum_{i=0}^{\infty} H_{i+1}^{(n)}x^i = \left(\frac{1+(P-1)x}{1-x-x^2}\right)^n = \sum_{i=0}^{\infty} F_{i+1}^{(n)}x^i \sum_{j=0}^{n} \binom{n}{j}(P-1)^j x^j \tag{8}$$

$$= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j+1}^{(n)}\right)x^i .$$

Hence,

$$H_{i+1}^{(n)} = \sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j+1}^{(n)} . \tag{9}$$

Using (5) together with the fact that $\binom{n}{j} = 0$ for $j > n$, we have

$$\lim_{i\to\infty}\frac{H_{i+1}^{(n)}}{F_{i-n}^{(n)}} = \lim_{i\to\infty}\sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j+1}^{(n)}/F_{i-n}^{(n)} = \sum_{j=0}^{n}\binom{n}{j}(P-1)^j \alpha^{n-j+1}$$

$$= \alpha \lim_{i\to\infty}\sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j}^{(n)}/F_{i-n}^{(n)} = \alpha \lim_{i\to\infty}\frac{H_i^{(n)}}{F_{i-n}^{(n)}}$$

so that

$$\lim_{i\to\infty}\frac{H_{i+1}^{(n)}}{H_i^{(n)}} = \alpha \tag{10}$$

and

$$\lim_{i\to\infty}\frac{H_{i+k}^{(n)}}{H_{i+m}^{(n)}} = \alpha^{k-m} . \tag{11}$$

By (6) and an argument similar to that used in the derivation of (10), we have

$$\lim_{i\to\infty}\frac{H_i^{(n)}}{F_{i-n}^{(n+1)}} = 0$$

while

$$\lim_{i\to\infty}\frac{H_i^{(n+1)}}{F_{i-n}^{(n+1)}} = \sum_{j=0}^{n+1}\binom{n+1}{j}(P-1)^j\alpha^{n-j} \neq 0$$

so that

$$\lim_{i\to\infty}\frac{H_i^{(n)}}{H_i^{(n+1)}} = 0 \tag{12}$$

and

$$\lim_{i\to\infty}\frac{H_{i+k}^{(n)}}{H_{i+m}^{(n+1)}} = 0 . \tag{13}$$

Let $R_{(n)}^*(x)$ be the generating function for the sequence of elements in the n^{th} row of the convolution array formed by the powers of $C_1^*(x)$. Then

(14) $$R_n^*(x) = \sum_{i=0}^{\infty} H_n^{(i+1)} x^i .$$

In [1], it is shown that

(15) $$R_1^*(x) = (1-x)^{-1}$$

(16) $$R_2^*(x) = P(1-x)^{-2}$$

and

(17) $$R_n^*(x) = \frac{(1+(P-1)x)N_{n-1}^*(x) + (1-x)N_{n-2}^*(x)}{(1-x)^n} = \frac{N_n^*(x)}{(1-x)^n}, \quad n \geq 3 ,$$

where $N_n^*(x)$ is a polynomial of degree $n-2$ for $n \geq 2$.

Let $G_n^*(x)$ be the generating function for the n^{th} column of the left-adjusted triangular array formed by the coefficients of the $N_n^*(x)$ polynomials. In [1], it is shown that

(18) $$G_1^*(x) = C_1^*(x)$$

(19) $$G_2^*(x) = DC_2(x)$$

and

(20) $$G_n^*(x) = \frac{(P-1-x)}{(1-x-x^2)} G_{n-1}^*(x), \quad n \geq 3 ,$$

where $D = P^2 - P - 1$. By induction, it can be shown that

(21) $$G_n^*(x) = \frac{(P-1-x)^{n-2}}{(1-x-x^2)^n}, \quad n \geq 3$$

which by an argument similar to that of (8) yields

(22) $$G_n^*(x) = D \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} (-1)^j \binom{n-2}{j} (P-1)^{n-j-2} F_{i-j+1}^{(n)} \right) x^i .$$

If we let $g_{i+1}^{(n)}$ be the coefficient of x^i in $G_n^*(x)$ then we see that

(23) $$g_{i+1}^{(1)} = F_{i+1} + (P-1)F_i$$

(24) $$G_{i+1}^{(2)} = DF_{i+1}^{(2)}$$

and

(25) $$g_{i+1}^{(n)} = D \sum_{j=0}^{i} (-1)^j \binom{n-2}{j} (P-1)^{n-j-2} F_{i-j+1}^{(n)}, \quad n \geq 3.$$

Following arguments similar to those given in obtaining (10) through (13), we have

(26) $$\lim_{i \to \infty} \frac{g_{i+1}^{(n)}}{g_i^{(n)}} = \alpha$$

(27) $$\lim_{i \to \infty} \frac{g_{i+k}^{(n)}}{g_{i+m}^{(n)}} = \alpha^{k-m}$$

(28) $$\lim_{i \to \infty} \frac{g_i^{(n)}}{g_i^{(n+1)}} = 0$$

and

(29) $$\lim_{i \to \infty} \frac{g_{i+k}^{(n)}}{g_{i+m}^{(n+1)}} = 0.$$

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SUMMATION OF MULTIPARAMETER HARMONIC SERIES

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1. INTRODUCTION

Consider the multiparameter alternating harmonic series denoted and defined by

(1) $$\omega(j; k_1, \cdots, k_n) = \sum_{i=0}^{\infty} (-1)^i/(j+s_i),$$

where j and the k_i are positive integers, $s_0 = 0$, $s_n = S$, and

$$s_i = [i/n]S + \sum_{t=1}^{i, \text{mod } n} k_t.$$

Note that the parameters $k_1, \cdots, k_n$ are successive cyclic denominator increments. In the ensuing treatment summation formulas for such series, to be called ω-series, are developed which admit evaluation in terms of elementary functions. An example is included to illustrate the formulas.

2. SUMMATION FORMULAS

The expression of the summation formulas for the ω-series (1) is based upon the following two lemmas.

Lemma 1.

(2) $$\omega(j; k) = (\tfrac{1}{2}k)G(j/k) = \int_0^1 x^{j-1}dx/(1+x^k)$$

$$= (-1)^{j-1}(r/k)\ln(1+x)$$

$$-(2/k)\sum_{i=0}^{q-1} [P_i(x)\cos((2i+1)j\pi/k) - Q_i(x)\sin((2i+1)j\pi/k)]\Big|_0^1,$$

[Continued on page 144.]