AN APPLICATION OF TRIBONACCI NUMBERS

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An interesting application of the Tribonacci numbers appeared unexpectedly in the solution of the following problem. Begin with 4 nonnegative integers, for example, 9, 4, 6, 7. Take cyclic differences of pairs of numbers (the smaller number from the larger) where the fourth difference is always the difference between the last number (7 in the above example) and the first number (9 in the above example). Repeat this process on the differences. For the example above, we have

$$\begin{array}{lcccccccccc}
1^{st}\text{ row} & 9 & & 4 & & 6 & & 7 & & & \\
2^{nd}\text{ row} & & 5 & & 2 & & 1 & & 2 & & \\
3^{rd}\text{ row} & & & 3 & & 1 & & 1 & & 3 & \\
4^{th}\text{ row} & & & & 2 & & 0 & & 2 & & 0 \\
5^{th}\text{ row} & & & & & 2 & & 2 & & 2 & & 2 \\
6^{th}\text{ row} & & & & & & 0 & & 0 & & 0 & & 0.
\end{array}$$

Starting with the numbers 9, 4, 6, 7 and following the procedure described, the process terminates in the 6^{th} row with all zeros.

Problem. Are there 4 starting numbers that will terminate with all zeros in the 7^{th} row, the 8^{th} row, $\cdots$, the n^{th} row?

Various sequences of numbers were tried but they were found unsatisfactory. One development that leads to a solution is outlined below.

(a) Begin with 4 numbers, not all zero,

$$a \quad b \quad c \quad d \tag{1}$$

which are assumed to be known and then try to get the 4 numbers in the row directly above *a, b, c, d,* namely, the numbers

$$x_1 \quad x_2 \quad x_3 \quad x_4\,. \tag{2}$$

Thus,

$$\begin{array}{lcccc}
2^{nd}\text{ row} & x_1 & x_2 & x_3 & x_4 \\
1^{st}\text{ row} & a & b & c & d
\end{array} \tag{3}$$

(b) Now, rather than try to solve the problem for arbitrary numbers *a, b, c, d,* we will take the special case where

$$d = a + b + c. \tag{4}$$

In place of (3), we have

$$\begin{array}{lcccc}
2^{nd}\text{ row} & x_1 & x_1 + a & x_1 + a + b & x_1 + a + b + c \\
1^{st}\text{ row} & a & b & c & d = a + b + c.
\end{array} \tag{5}$$

At this point, one can select x_1 to be any nonnegative integer. However, this procedure proves rather unproductive. We now assume that the summability pattern for the 4 known starting numbers

$$a \quad b \quad c \quad d = a + b + c$$

also holds for

$$x_1 \quad x_1 + a \quad x_1 + a + b \quad x_1 + a + b + c. \tag{6}$$

;r the above assumption, we have

$$x_1 + (x_1 + a) + (x_1 + a + b) = x_1 + a + b + c.$$

/ing for x_1 , we get

$$x_1 = \frac{c-a}{2},$$

:re now x_1 is determined in terms of the known numbers a and c. Note that $c - a$ must be even for x_1 to be nteger.

) For a given set of 4 numbers $a, b, c, d = a + b + c$, once x_1 is determined, we can get the 2^{nd} row in (5). umably, the procedure can then be repeated on the 2^{nd} row to get a 3^{rd}, 4^{th}, etc. row. The following ex- ıle shows that another slight modification is necessary.

‹ample 1. Begin with the four numbers 1, 1, 1, 3. These numbers satisfy the summability condition $a + b + c$. Using the condition in (8) with $a = 1, c = 1$, we have

$$x_1 = \frac{c-a}{2} = 0.$$

stituting in (5), we get

$$\begin{array}{lccccccccc} 2^{nd}\text{ row} & 0 & & 1 & & 2 & & 3 & & \\ 1^{st}\text{ row} & & 1 & & 1 & & 1 & & 3. \end{array}$$

2^{nd} row now serves as our 4 known numbers $a, b, c, d = a + b + c$. Here $a = 0, c = 2$ and from (8), we have

)
$$x_1 = \frac{c-a}{2} = 1$$

ng (5) and (9), we now have

ı)
$$\begin{array}{lccccccccc} 3^{rd}\text{ row} & \boxed{1 \quad 1 \quad 2 \quad 4} \\ 2^{nd}\text{ row} & \quad 0 \quad 1 \quad 2 \quad 3 \\ 1^{st}\text{ row} & \qquad 1 \quad 1 \quad 1 \quad 3. \end{array}$$

Ve now go on to the 4^{th} row. However, if we take the 3^{rd} row 1, 1, 2, 4 in (11) as our 4 known numbers, :n $a = 1, c = 2$ and from (8)

!)
$$x_1 = \frac{c-a}{2} = \frac{1}{2}$$

ıich is not an integer. Apparently, we cannot get the 4^{th} row from our present method.

Ve pause to point out several items of interest in the example above.

l. We began the example 1 with the 4 starting numbers 1, 1, 1, 3. This was a rather arbitrary selection. If we ǀ started with the 4 numbers 0, 0, 2, 2 we could have calculated the 4^{th} row but the numbers here would /e been 1, 1, 2, 4 precisely the same as in our present example where again we would have been stopped. :re appears to be no marked advantage in selecting other starting numbers rather than 1, 1, 1, 3.

. In (11) the numbers in the 3^{rd} row are the first four numbers of the classical Tribonacci sequence

)
$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ T_1 & T_2 & T_3 & T_4 . \end{array}$$

If we start with the Tribonacci numbers in (13), we have for the cyclic differences

1.	1	1	2	4
2.	0	1	2	3
3.	1	1	1	3
4.	0	0	2	2
5.	0	2	0	2
6.	2	2	2	2
7.	0	0	0	0

·t all zeros in the seventh row.

Let us now return to (11) where our procedure was stopped. Multiply each element in each row of (11) by 2. We have

$$\begin{array}{lcccccccc} & 3^{rd}\text{ row} & 2 & & 2 & & 4 & & 8 \\ (15) & 2^{nd}\text{ row} & & 0 & & 2 & & 4 & & 6 \\ & 1^{st}\text{ row} & & & 2 & & 2 & & 2 & & 6\,. \end{array}$$

In the third row of (15), $a = 2$, $c = 4$ and using (8), we have

$$(16) \qquad x_1 = \frac{c-a}{2} = 1$$

We can now get the 4^{th} row. From the 4^{th} row, we can get the 5^{th} row and from the 5^{th} row, we can get the 6^{th} row before we are stopped by a non-integral value of x_1. The cyclic differences are shown below.

$$\begin{array}{llcccccccccc} & 6. & \boxed{2 \quad 4 \quad 7 \quad 13} \\ & 5. & 2 & 3 & 6 & 11 \\ & 4. & & 1 & 3 & 5 & 9 \\ (17) & 3. & & & 2 & 2 & 4 & 8 \\ & 2. & & & & 0 & 2 & 4 & 6 \\ & 1. & & & & & 2 & 2 & 2 & 6 \end{array}$$

As in (11) so in (17), the four numbers in row 6 (where we are stopped) are consecutive Tribonacci numbers T_3 to T_6. A list of the first seventeen Tribonacci numbers is given below.

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad n = 4, 5, 6, \cdots.$$
$$T_1 = T_2 = 1$$
$$T_3 = 2.$$

(18)

1	1	2	4	7	13	24	44	81	149
T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
274	504	927	1705	3136	5768	10,609			
T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}			

If we return to (17) and multiply each element in each row by 2, we can get rows 7, 8, 9 before we are stopped. The 4 numbers in row 9 are the 4 Tribonacci numbers 7, 13, 24, 44 (T_5 to T_8, see (18)).

The procedure is now clear. From (11), (15) and (17), whenever we are stopped, we multiply each element in each row by 2. This will allow us to go 3 rows upward. We are then stopped at a set of 4 Tribonacci numbers where the first two Tribonacci numbers overlap with the last two Tribonacci numbers of the preceding stopping point. If in (11) and (17), we take the cyclic differences from row 1 downward, we get 4 more rows before terminating in all zeros. We summarize the results.

(19)

Starting Tribonacci numbers	Rows upward counting from		Rows downward not counting		Total rows
T_1 to T_4	row 1, 1, 1, 3	3	row 1, 1, 1, 3	4	7
T_3 to T_6	row 2, 2, 2, 6	6	row 2, 2, 2, 6	4	10
T_5 to T_8	row 4, 4, 4, 12	9	row 4, 4, 4, 12	4	13
T_7 to T_{10}	row 8, 8, 8, 24,	12	row 8, 8, 8, 24	4	16
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T_{2n+1} to T_{2n+4}	row $2^n, 2^n, 2^n, (3)2^n$,	$3(n+1)$	row $2^n, 2^n, 2^n, (3)2^n$,	4	$3(n+2)+1$

where $n = 0, 1, 2, 3, \cdots$.

If we take the four consecutive Tribonacci numbers T_{2n+1} to T_{2n+4}, $n = 0, 1, 2, 3, \cdots$ we get all zeros in the $3(n+2)+1$ row.

The starting Tribonacci numbers above begin with an odd-numbered term such as T_1, T_3, T_5, and so on. What happens if we start with an even-numbered term of the sequence, say T_2, T_4, T_6, and so on? Actually, we get all zeros at precisely the same row as we did when we started with the odd-numbered Tribonacci sequence T_1, T_3, T_5, and so on. The summary is given below.

	Starting Tribonacci numbers	Rows upward counting from		Rows downward not counting		Total rows
(20)	T_2 to T_5	row 1, 1, 3, 5	3	row 1, 1, 3, 5,	4	7
	T_4 to T_7	row 2, 2, 6, 10	6	row 2, 2, 6, 10	4	10
	T_6 to T_9	row 4, 4, 12, 20	9	row 4, 4, 12, 20	4	13
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	T_{2n} to T_{2n+3}	$2^{n-1}, 2^{n-1}, (3)2^{n-1}, (5)2^{n-1}$	$3n$	(see column 2)	4	$3(n+1)+1$

where $n = 1, 2, 3, \cdots$.

We can rewrite the results in (19) to agree with the values of n in (20). Thus, for $n = 1, 2, 3, \cdots$

(21) Odd numbered starting Tribonacci numbers $\quad T_{2n-1},\quad T_{2n},\quad T_{2n+1},\quad T_{2n+2}$

(22) Even numbered starting Tribonacci numbers $\quad T_{2n},\quad T_{2n+1},\quad T_{2n+2},\quad T_{2n+3}$

will give all zeros for the $3(n+1)+1$ row.

Conclusion. What are 4 starting numbers which give all zeros at precisely row m, where $m = 1, 2, 3, \cdots$?

	Number of rows for which we get all zeros	4 starting numbers
(23)	$m = 1$	0, 0, 0, 0
	$m = 2$	1, 1, 1, 1
	$m = 3$	2, 0, 2, 4
	$m = 4$	0, 2, 2, 4
	$m = 5$	1, 1, 3, 5

For $m \geqslant 6$, note that the numbers 6, 7, 8, 9, $\cdots$, are

a. multiples of 3, so that $m = 3(n+1)$, $\quad n = 1, 2, 3, \cdots$,
b. multiples of 3 plus 1, so that $m = 3(n+1)+1$, $\quad n = 1, 2, 3, \cdots$,
c. multiplies of 3 plus 2, so that $m = 3(n+1)+2$, $\quad n = 1, 2, 3, \cdots$.

Actually, we have already solved the problem for the case where $m = 3(n+1)+1$, $n = 1, 2, 3, \cdots$ (for m equal to a multiple of 3 plus 1) in (21) and (22). If we take the solution (21), we can easily get the row above (21) which will be the solution for $m = 3(n+1)+2$, $n = 1, 2, 3, \cdots$. Moreover, if we go downward from (21) by taking the cyclic differences, we will have the solution for the case $m = 3(n+1)$, $n = 1, 2, 3, \cdots$. Thus,

		Starting Tribonacci Numbers				Solution for
(24)	Upward from (21)	0	T_{2n-1}	$T_{2n+1}+T_{2n}$	T_{2n+2}	$m = 3(n+1)+2$
	Relation (21)	T_{2n-1}	T_{2n}	T_{2n+1}	T_{2n+2}	$m = 3(n+1)+1$
(25)	Downward from (21)	$T_{2n}-T_{2n-1}$	$T_{2n+1}-T_{2n}$	$T_{2n+2}-T_{2n+1}$	$T_{2n+2}-T_{2n-1}$	$m = 3(n+1)$

Example 2. Find the 4 starting numbers that give all zeros for precisely the 8^{th} row.

Solution. Here $m = 8$ and m is a multiple of 3 plus 2. From $m = 3(n+1)+2$ we have $8 = 3(n+1)+2$ or $n = 1$. From (24) the 4 starting Tribonacci numbers are $0, T_1, T_1+T_2, T_4$ and concretely from (18) 0, 1, 2, 4.

Now

(26)

$$\begin{array}{lcccc}
1. & 0 & 1 & 2 & 4 \\
2. & 1 & 1 & 2 & 4 \\
3. & 0 & 1 & 2 & 3 \\
4. & 1 & 1 & 1 & 3 \\
5. & 0 & 0 & 2 & 2 \\
6. & 0 & 2 & 0 & 2 \\
7. & 2 & 2 & 2 & 2 \\
8. & 0 & 0 & 0 & 0
\end{array}$$

Using (21), (24) and (25) we have constructed the following table.

Table

m	n	4 Starting Tribonacci Numbers
6	1	0, 1, 2, 3
7	1	1, 1, 2, 4
8	1	0, 1, 2, 4
9	2	2, 3, 6, 11
10	2	2, 4, 7, 13
11	2	0, 2, 6, 13
12	3	6, 11, 20, 37
13	3	7, 13, 24, 44
14	3	0, 7, 20, 44

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[Continued from page 116.]

where

$$q = [k/2], \quad r = k, \bmod 2, \quad 1 \leqslant j \leqslant k,$$

$$P_i(x) = (\tfrac{1}{2}) ln [x^2 - 2x \cos((2i+1)\pi/k) + 1],$$

$$Q_i(x) = \arctan [(x - \cos((2i+1)\pi/k)/\sin((2i+1)\pi/k)] .$$

Proof. The G function has the series and integral representation [4, p. 20]

$$G(z) = 2 \sum_{n=0}^{\infty} (-1)^n/(z+n) = 2 \int_0^1 x^{z-1} dx/(1+x)$$

from which the first part of (2) is immediate. The integration formula is recorded in [5, p. 20].

Lemma 2.

(3) $$\omega(j; k_1, k_2) = (1/S)[\psi((j+k_1)/S) - \psi(j/S)] ,$$

where the psi (digamma) function is the logarithmic derivative of the gamma function and has integral representation for rational argument u/v, $0 < u < v$,

(4) $$\begin{aligned} \psi(u/v) &= -C + v \int_0^1 (x^{v-1} - x^{u-1}) dx/(1-x^v) \\ &= -C - ln v - (\pi/2) \cot(u\pi/v) \\ &\quad + \sum_{i=1}^{q} \cos(2ui\pi/v) ln(4 \sin^2 i\pi/v) + (-1)^u \delta_0^r ln 2 , \end{aligned}$$

where $q = [(v-1)/2]$, $r = u/2 - [u/2]$, C is Euler's constant.

[Continued on page 149.]