DIOPHANTINE EQUATIONS
INVOLVING THE GREATEST INTEGER FUNCTION

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It is known [1, p. 142] that if λ and μ are fixed positive irrationals such that $\mu\lambda = \mu + \lambda$, then the equation $[n\lambda] = [m\mu]$ has no solution in integers m and n, where $[x]$ denotes the greatest integer less than or equal to x. We prove the following generalization.

Theorem. Let λ and μ be fixed positive irrationals. The equation $[n\lambda] = [m\mu]$ has no solution in integers m and n if and only if $\mu\lambda = b\mu + c\lambda$ for some integers b and c such that $\lambda > b > 0$.

Proof. Let $\boldsymbol{Z}$ denote the set of integers. Suppose first that $\mu\lambda = b\mu + c\lambda$, where $b,c \in \boldsymbol{Z}$, $\lambda > b > 0$. Assume (for the purpose of contradiction) that

$$[n\lambda] = [m\mu] \tag{1}$$

for some $m,n \in \boldsymbol{Z}$. Write $\theta = \mu/\lambda$, $\epsilon = m\theta - [m\theta]$. Since $\mu = b\theta + c$, θ is irrational and thus $0 < \epsilon < 1$. By (1), $n\lambda = m\mu + \sigma$, where $-1 < \sigma < 1$. Thus $n = m\theta + \sigma/\lambda = [m\theta] + (\epsilon + \sigma/\lambda)$. Since $\lambda > 1$, $-1 < (\epsilon + \sigma/\lambda) < 2$. Therefore, $n = [m\theta] + \delta$, where $\delta = 0$ or 1.

We have

$$m\mu = mb\theta + mc = b\epsilon + b[m\theta] + mc. \tag{2}$$

Hence,

$$[m\mu] = [b\epsilon] + b[m\theta] + mc. \tag{3}$$

We have, using (2),

$$\begin{aligned}[n\lambda] &= [(m\theta + \delta - \epsilon)\lambda] = [m\mu + (\delta - \epsilon)\lambda] = [b\epsilon + b[m\theta] + mc + (\delta - \epsilon)\lambda] \\ &= [b\epsilon + (\delta - \epsilon)\lambda] + b[m\theta] + mc.\end{aligned} \tag{4}$$

Since the left sides of (3) and (4) are equal,

$$[b\epsilon] = [b\epsilon + (\delta - \epsilon)\lambda].$$

If $\delta = 0$, then $[b\epsilon] = [(b - \lambda)\epsilon]$, a contradiction, since $b\epsilon > 0$ and $(b - \lambda)\epsilon < 0$. If $\delta = 1$, then

$$b > [b\epsilon] = [b\epsilon + (1 - \epsilon)\lambda] \geqslant [b\epsilon + (1 - \epsilon)b] = b,$$

a contradiction. This proves that there are no integers m,n for which (1) holds.

To prove the converse, it suffices to show that (1) has a solution in each of the following three cases. Case 1: μ, θ, and 1 are linearly independent over the rationals, i.e., if $a\mu\lambda = b\mu + c\lambda$ with $a,b,c \in \boldsymbol{Z}$, then $a = b = c = 0$; Case 2: $a\mu\lambda = b\mu + c\lambda$, where a, b, and c are relatively prime integers, $a \geqslant 0$, and $a \neq 1$; Case 3: $\mu\lambda = b\mu + c\lambda$, where $b,c \in \boldsymbol{Z}$ and either $b < 0$ or $\lambda < b$.

Case 1. By Kronecker's Theorem [2, p. 382], there exist $m, z_1, z_2 \in \boldsymbol{Z}$ such that

$$m\mu = 1/2 + z_1 + E_1$$

and

$$m\theta = 1/3(1 + \lambda) + z_2 + E_2,$$

where $|E_i| < 1/6(1 + \lambda)$ for $i = 1,2$. Then

$$\epsilon = m\theta - [m\theta] = 1/3(\lambda + 1) + E_2$$

and

$$m\mu - \epsilon\lambda = (1/2 - \lambda/3(\lambda + 1)) + z_1 + (E_1 - \lambda E_2).$$

Since $|E_1 - \lambda E_2| < 1/6 < 1/2 - \lambda/3(\lambda + 1)$, we have $[m\mu - \epsilon\lambda] = z_1$. Since $[m\mu] = z_1$, we have

$$[m\mu] = [m\mu - \epsilon\lambda] = [(m\theta - \epsilon)\lambda] = [[m\theta]\lambda],$$

so that Eq. (1) has a solution with $n = [m\theta]$.

Case 2. If $a = 0$, then (1) has the solution $m = b$, $n = -c$. Thus assume $a \geqslant 2$. Since $(a,b,c) = 1$, either $a \nmid b$ or $a \nmid c$. Without loss of generality, we assume $a \nmid b$. Since $\mu = b\theta/a + c/a$, θ is irrational. Thus there exist $p,q \in \boldsymbol{Z}$ such that $p\theta = \eta + q + E$, where $\eta = 1/a + 1/2a(a\lambda + |b|)$ and $|E| < \eta - 1/a$. Let $m = ap$ and $\epsilon = m\theta - [m\theta]$. Then

$$m\theta = (aq + 1) + (a\eta - 1) + aE,$$

so that

(5) $$[m\theta] = aq + 1.$$

Also, $\epsilon = (a\eta - 1) + aE$, so that

(6) $$0 < \epsilon < 2(a\eta - 1) = 1/(a\lambda + |b|).$$

By (5),

(7) $$m\mu = mb\theta/a + mc/a = b\epsilon/a + b[m\theta]/a + mc/a = b\epsilon/a + b/a + bq + pc.$$

Thus,

(8) $$[m\mu] = [b\epsilon/a + b/a] + bq + pc.$$

Since $b \nmid a$ and since $|b\epsilon/a| < 1/a$ by (6), it follows from (8) that

(9) $$[m\mu] = [b/a] + bq + pc.$$

By (7),

$$m\mu - \epsilon\lambda = (b - a\lambda)\epsilon/a + b/a + bq + pc,$$

so that

(10) $$[m\mu - \epsilon\lambda] = [(b - a\lambda)\epsilon/a + b/a] + bq + pc.$$

Since $|(b - a\lambda)\epsilon/a| < 1/a$ by (6), it follows from (10) that

(11) $$[m\mu - \epsilon\lambda] = [b/a] + bq + pc.$$

By (9) and (11),

$$[m\mu] = [m\mu - \epsilon\lambda] = [(m\theta - \epsilon)\lambda] = [[m\theta]\lambda] \quad .$$

Thus (1) has a solution with $n = [m\theta]$.

Case 3. We argue as in Case 2 with $a = 1$. By (8) with $a = 1$,

(12) $$[m\mu] = [b\epsilon] + b + bq + pc.$$

By (10) with $a = 1$,

(13) $$[m\mu - \epsilon\lambda] = [(b - \lambda)\epsilon] + b + bq + pc.$$

By (6), with $a = 1$, $0 < \epsilon < 1/(\lambda + |b|)$. Thus $|b\epsilon| < 1$ and $|(b - \lambda)\epsilon| < 1$. Moreover, by the hypotheses of Case 3, $b\epsilon$ and $(b - \lambda)\epsilon$ have the same sign. Thus, by (12) and (13),

$$[m\mu] = [m\mu - \epsilon\lambda] = [(m\theta - \epsilon)\lambda] = [[m\theta]\lambda].$$

Therefore (1) has a solution with $n = [m\theta]$. Q.E.D.

Corollary 1. Let λ be a positive irrational. Then $[n\lambda] = [m\lambda^2]$ has no solution with $n,m \in \boldsymbol{Z}$ if and only if $\lambda = (b + (b^2 + 4c)^{1/2})/2$ for some positive integers b and c.

Proof. Note that if $\mu\lambda = b\mu + c\lambda$ with $b,c \in \boldsymbol{Z}$ and $\lambda > b > 0$, then $(\lambda - b)(\mu - c) = bc$, so that $c > 0$. Hence Corollary 1 follows from the Theorem with $\mu = \lambda^2$. Q.E.D.

Corollary 2. Let λ be a positive irrational. Then $[n\lambda] = [m\lambda] + m$ has no solution with $n,m \in \boldsymbol{Z}$ if and only if

$$\lambda = ((b + c - 1) + ((b + c - 1)^2 + 4b)^{1/2})/2$$

for some positive integers b and c.

Proof. This follows from the Theorem with $\mu = \lambda + 1$.

Corollary 3. Let σ be a positive irrational. Then $[n\sigma] + n = [m/\sigma] + m$ has no solution with $n,m \in \boldsymbol{Z}$.

Proof. This follows from the Theorem with $\mu = 1 + 1/\sigma$, $\lambda = \sigma + 1$, and $b = c = 1$. Q.E.D.

(Corollary 3 is part of Problem 22 in [3, p. 84].)

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For an ω-series with an arbitrary odd number of k_i parameters two cycles of parametric incrementation are required to bring the series into alignment for grouping. Use of the identity

$$G(z) = \psi(z/2 + 1/2) - \psi(z/2),$$

[4, p. 20], and Lemma 1 render the following summation expression.

Theorem 2.

$$\omega(j; k_1, \cdots, k_{2n+1}) = \sum_{i=0}^{2n} (-1)^i \omega(j + s_i ; S) = (1/2S) \sum_{i=0}^{2n} (-1)^i G((j + s_i)/S).$$

3. EXAMPLES

Some calculations for the uniparameter ω-series are to be found in [1] and for the biparameter series in [2]. The above theorems and their proofs can be illustrated with the following triparameter ω-series:

$$\begin{aligned}
\omega(1; 1, 1, 2) &= [(1 - 1/2) + (1/3 - 1/5) + (1/6 - 1/7)] + [(1/9 - 1/10) + (1/11 - 1/13) + \cdots] \\
&\qquad + [(1/17 - 1/18) + \cdots] + \cdots \\
&= (1 - 1/2) + (1/9 - 1/10) + (1/17 - 1/18) + \cdots + (1/3 - 1/5) + (1/11 - 1/13) + \cdots \\
&\qquad + (1/6 - 1/7) + \cdots \\
&= \omega(1; 1, 7) + \omega(3; 2, 6) + \omega(6; 1, 7) \\
&= (1/8)[G(3/4) - G(1/2) + G(1/4)] \\
&= (1/8)[\sqrt{2}(\pi - 21n(1 + \sqrt{2}) - \pi + \sqrt{2}(\pi + 21n(1 + \sqrt{2}))] \\
&= (\pi/8)[2\sqrt{2} - 1].
\end{aligned}$$

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