ON FIBONACCI AND TRIANGULAR NUMBERS

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The infinite sequence discovered by the author in [1], namely the numerators of C_k, i.e.,

(1) $$F_{2k}C_k = (1 + L_k + F_{2k-1})$$

are related to the Triangular numbers $\{T_n\}$, where $T_{-1} = 0 = T_0$ and

(2) $$T_n = n(n+1)/2 \quad \text{for all integral } n,$$

in general. It is interesting that four members of the sequence defined by T_{-1+F_n} are zero, namely those for $n = -1, 0, 1, 2$. It will be shown that

(3) $$F_{2k}C_k = T_{1+F_{k+1}} + T_{-1+F_{k-2}}$$

for all natural numbers k. The first term on the right-hand side merely picks off the 2, 3, 4, 6, 9th ... terms of $\{T_n\}$.

Proof. The proof is direct and easy considering that (3) is not obvious. We first need

(4) $$3F_{k+1} - F_{k-2} = 2L_k$$

which is easily derived from $F_{k+1} + F_{k-1} = L_k$. Next we need

$$F_{k+1}^2 = F_{2k} + F_{k-1}^2 \quad \text{and} \quad F_{k-1}^2 = F_{2k-3} - F_{k-2}^2$$

which are (I_{10}) and (I_{11}) of Hoggatt [2] which enables us to write

(5) $$F_{k+1}^2 + F_{k-2}^2 = 2F_{2k-1}\ .$$

First we write

$$2T_{1+F_{k+1}} + 2T_{-1+F_{k-2}} = (1+F_{k+1})(2+F_{k+1}) + (-1+F_{k-2})F_{k-2} = 2 + 3F_{k+1} + F_{k+1}^2 + F_{k-2}^2 - F_{k-2}$$

which via (4) and (5) $$= 2 + 2L_k + 2F_{2k-1}$$

as was to be shown.

Table of C_kF_{2k} Numbers and Triangular Numbers

k	0	1	2	3	4	5	6	7	8	9	10	11	12
C_kF_{2k}	4	3	6	10	21	46	108	263	658	1674	4305	11146	28980
$T_{1+T_{k+1}}$	3	3	6	10	21	45	105	253	630	1596	4095	10585	27495
$T_{-1+F_{k-2}}$	1	0	0	0	0	1	3	10	28	78	210	561	1485

Now it would be nice if a generalization obtained for the generalized $C_{j,k}$ in the author's second paper on sums of Fibonacci reciprocals [3]. Such *is* the case. First we must define generalized Triangular numbers

(6) $$T_{n,j} = n(n+j)/2$$

which may *not* always be integers. Let $\{P_n\}$ be any generalized sequence such that

(7) $$P_{n+1} = jP_n + P_{n-1}\,,$$

where j is an integer; then using the general Binet formula one can show that

(8) $$P_{2n+1} = P_{n+1}^2 + P_n^2$$

and it definitely is equally obvious that we can show

(9) $$jP_{2n} = P_{n+1}^2 - P_{n-1}^2 \; .$$

Using (8) and (9), we may show that

(10) $$P_{k+1}^2 + P_{k-2}^2 = jP_{2k} + P_{2k-3} = (j^2+1)P_{2k-1}$$

which corresponds to (5) in the Fibonacci case. Now the author [3, (9)] has shown that the numerators of C_{jk} are

(11) $$P_{2k}C_{j,k} = (1 + P_k^* + P_{2k-1}) \; .$$

The j subscript has been dropped from the P's for neatness but they are still a function of j and ideally we should write $P_{j,k}$,

Theorem.

(12) $$(1 + P_k^* + P_{2k-1}) = (1 + 2T_{P_{k,j}} + 2T_{P_{k-2,2}}).$$

The proof is straightforward and note that $P_k^* = P_{k+1} + P_{k-1}$ is by definition the Lucas complement of P_k. From (6) Eq. (12) becomes

(13) $$(1 + P_k(P_k + j) + P_{k-1}(P_{k-1} + 2) = (1 + jP_k + 2P_{k-1} + P_k^2 + P_{k-1}^2) = (1 + P_{k+1} + P_{k-1} + P_{2k-1})$$

by using (8). Note that we did not use (9) and that has led to (12) being different from (3). I illustrate this by taking $C_{3,4}$ = 1309/3927. Now $\{P_{3,k}\}$ is 0, 1, 3, 10, 33, 109, 360, 1189, $\cdots$. According to (11) and (12) the numerator of $C_{3,4}$ is 1 + 33(33 + 3) + 10(10 + 2) = 1309 as it should. In (12) be careful to note that j and 2 are subscripts of T and not of P.

H. W. Gould has called my attention to a known theorem [4] that an integer m is the sum of two triangular numbers if and only if $4m + 1$ is the sum of two squares, say $4m + 1 = u^2 + v^2$, where $(u - v) \geqslant 3$. Hence for the sequence $G_k = C_kF_{2k}$ we have the following table.

k	$(1 + 4C_kF_{2k}) = (u^2 + v^2)$	k	$(1 + 4C_kF_{2k}) = (u^2 + v^2)$
0	$17 = 4^2 + 1^2$	5	$185 = 8^2 + 11^2$
1	$13 = 2^2 + 3^2$	6	$433 = 12^2 + 17^2$
2	$25 = 4^2 + 3^2$	7	$1053 = 18^2 + 29^2$
3	$41 = 4^2 + 5^2$	8	$2633 = 28^2 + 43^2$
4	$85 = 6^2 + 7^2$	9	$6697 = 44^2 + 69^2$

We noticed that the differences between adjacent u numbers seems to be twice the Fibonacci numbers and that a similar relation holds for the v numbers. V. E. Hoggatt, Jr., in a letter dated Jan. 22, 1977, has found the following closed form.

(14) $$1 + 4G_k = 1 + 4C_kF_{2k} = (2(1 + F_{k-1}))^2 + (1 + 2F_k)^2 = u^2 + v^2 \; .$$

Now Sloane [5] contains the sequence $N^2 + (N - 1)^2$, his No. 1567, which generates a lot of primes. The sequence above may also be prime rich since 17, 13, 41, 433, 2633 are primes. Also G numbers for negative k values may be found in the recently submitted [6]. Then the sequence $(1 + 4G_{-k})$ for k = 0, 1, 2, $\cdots$ gives: 17, 9, 37, 41, 169, 317, 1009, 2329, 6581, $\cdots$ all of which are primes but 2329 and the perfect squares 9 and 169.

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