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PHI AGAIN: A RELATIONSHIP BETWEEN THE GOLDEN RATIO AND THE LIMIT OF A RATIO OF MODIFIED BESSEL FUNCTIONS

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In his study of infinite continued fractions whose partial quotients form a general arithmetic progression, D. H. Lehmer derived a formula for their evaluation in terms of modified Bessel Functions [1]. We have

$$F(a,b) = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots = [a_0, a_1, a_2, \cdots], \tag{1}$$

where $a_n = an + b$. It was shown that

$$F(a,b) = \frac{I_{\alpha-1}(2/a)}{I_\alpha(2/a)}, \tag{2}$$

where $\alpha = b/a$ and I_α is the modified Bessel function

$$I_\alpha(z) = i^{-\alpha} J_\alpha(iz) \sum_{m=0}^{\infty} \frac{(z/2)^{\alpha+2m}}{\Gamma(m+1)\Gamma(\alpha+m+1)}. \tag{3}$$

Using (1) and (2) with $c\alpha = 2/a$ and $b = c/2$, we have

$$F(a,b) = [b, a+b, 2a+b, \cdots] = \frac{I_{\alpha-1}(c\alpha)}{I_\alpha(c\alpha)}. \tag{4}$$

As $\alpha \to \infty$ $(a \to 0)$, in the limit (Theorem 5 of [1]),

$$\lim_{\alpha \to \infty} \frac{I_{\alpha-1}(c\alpha)}{I_\alpha(c\alpha)} = F(0,b) = [b, b, b, \cdots]. \tag{5}$$

But, for $b = 1$, $(c = 2)$, $F(0,1)$ is the positive root of the quadratic equation

$$1 + \frac{1}{x} = x \tag{6}$$

which is represented by the infinite continued fraction expansion $[1, 1, 1, \cdots]$.

[Continued on p. 152.]