THE PERIODIC GENERATING SEQUENCE

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Given an integer sequence \( S = \{a_1, a_2, \ldots \} \), \( a_i > 0 \). Form a new sequence \( \{r_n\} \) by first choosing two integers \( r_1 \) and \( r_0 \), then setting

\[
  r_m = r_{m-1}a_m + r_{m-2}, \quad a_m \in S.
\]

We call \( S \) a Generating Sequence.

Notice that for each \( r_k \in \{r_n\} \), we can reduce \( r_k \) to

\[
  r_k = A(k)r_0 + B(k)r_{-1},
\]

where \( A(k) \) and \( B(k) \) are integers.

Hence \( \{r_0, r_{-1}\} \) can be viewed as a "basis" for \( \{r_n\} \). Then,

\[
  r_{-1} = A(-1)r_0 + B(-1)r_{-1} \Rightarrow A(-1) = 0, \quad B(-1) = 1,
\]

\[
  r_0 = A(0)r_0 + B(0)r_{-1} \Rightarrow A(0) = 1, \quad B(0) = 0.
\]

**Theorem 1.** Suppose two sequences \( \{r'_n\} \) and \( \{r''_n\} \) are generated from the same sequence with different choices of \( r'_{-1}, r'_0 \) and \( r''_{-1}, r''_0 \), then

\[
  \begin{vmatrix}
    r'_{-1} & r'_0 \\
    r'_{-1} & r'_0
  \end{vmatrix} = (-1)^k
  \begin{vmatrix}
    r''_{-1} & r''_0 \\
    r''_{-1} & r''_0
  \end{vmatrix}.
\]

**Proof.** By induction.

**Notation:** Let

\[
  L = \begin{bmatrix}
    A(k) & B(k) \\
    A(k-1) & B(k-1)
  \end{bmatrix}.
\]

Notice that

\[
  \begin{bmatrix}
    r_k \\
    r_{k-1}
  \end{bmatrix} = L \begin{bmatrix}
    r_0 \\
    r_{-1}
  \end{bmatrix}.
\]

**Lemma.** \( \det(L) = (-1)^k \).

**Proof.**

\[
  \begin{vmatrix}
    r'_{-1} & r'_0 \\
    r'_{-1} & r'_0
  \end{vmatrix} = A(k-1)r'_0 + B(k-1)r'_{-1} = A(k)r''_0 + B(k)r''_{-1}
\]

\[
  \begin{vmatrix}
    r'_{-1} & r'_0 \\
    r'_{-1} & r'_0
  \end{vmatrix} = \begin{vmatrix}
    r''_{-1} & r''_0 \\
    r''_{-1} & r''_0
  \end{vmatrix} = \det(L) \begin{vmatrix}
    r''_{-1} & r''_0 \\
    r''_{-1} & r''_0
  \end{vmatrix} = \det(L) = (-1)^k.
\]

**Theorem 2.** Let

\[
  S = \{a_1, a_2, \ldots \}
\]

be the generating sequence for \( \{r_n\} \), then

\[
  A(m) = A(m-1)a_m + A(m-2)
\]

\[
  B(m) = B(m-1)a_m + B(m-2), \quad a_m \in S.
\]

**Proof.** We have
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\[ r_m = r_{m-1}a_m + r_{m-2} = [A(m)B(m)] \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = [A(m-1)B(m-1)] \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}a_m + [A(m-2)B(m-2)] \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} \]

\[ = [A(m)B(m)] = [A(m-1) + A(m-2)B(m-1) + 1a_m + B(m-2)]. \]

**Remark:** The above theorem shows that \{A(n)\} and \{B(n)\} are also sequences generated by \( S \). Recall that \( A(-1) = 0 \), \( A(0) = 1 \), \( B(-1) = 1 \), \( B(0) = 0 \).

We shall now investigate what happens when the generating sequence is an infinite periodic sequence \( P = \{a_1, \ldots, a_k\} \).

**Theorem 3.** If \( \{r_n\} \) is generated from \( P \), then

\[ [A(\nu + u)B(\nu + u)] = [A(\nu)B(\nu)]L^n. \]

**Proof.** Recall

\[ L = \begin{bmatrix} A(k) & B(k) \\ A(k-1) & B(k-1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix} = L \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}. \]

Then

\[ r_{\nu} = [A(\nu)B(\nu)] \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}, \]

\[ r_{\nu+u} = [A(\nu)B(\nu)] \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix} = [A(\nu)B(\nu)]L \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}, \]

\[ r_{2\nu+u} = [A(\nu)B(\nu)] \begin{bmatrix} r_{2k} \\ r_{2k-1} \end{bmatrix} = [A(\nu)B(\nu)]L^2 \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix} = [A(\nu)B(\nu)]L^2 \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}. \]

It is easy to see that

\[ r_{\nu+u} = [A(\nu)B(\nu)]L^n \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix} = [A(\nu + u)B(\nu + u)]L^n \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix} = [A(\nu)B(\nu)]L^n \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}. \]

**Corollary.**

\[ [A(\nu + u)B(\nu + u)] = (-1)^{\nu+k} [A(\nu)B(\nu)] \]

**Proof.** By Theorem 3, we get

\[ [A(\nu + u)B(\nu + u)] = [A(\nu)B(\nu)]L^n = [A(\nu + u)B(\nu + u)] = [A(\nu)B(\nu)]L^n \det(L^n). \]

**Theorem 4.** If a sequence \( \{r_n\} \) is generated from an infinite periodic sequence \( P \) with period \( k \), then

\[ r_{n+2k} - C(k) + (-1)^k r_{n} = 0, \]

where \( C(k) \) is a positive integer independent of the choice of \( r_{-1} \) and \( r_0 \).

**Proof.** Consider

\[ r_{n+2k} + x r_{n+k} + y r_n = 0. \]

Assume the theorem is true except for the existence of \( x \) and \( y \). We have
\[ r_{n+2k} + x r_{n+k} + yr_n = 0 = \{ [A(n + 2k)B(n + 2k)] + x[A(n + k)B(n + k)] + y[A(n)B(n)] \} \left( \begin{array}{c} r_0 \\ r_{-1} \end{array} \right) = 0 \]

These are solvable iff

\[ D = \begin{vmatrix} A(n + k) & B(n + k) \\ A(n) & B(n) \end{vmatrix} \neq 0. \]

Then by Theorem 3,

\[ [A(n + k)B(n + k)] = [A(n)B(n)]L = [A(n)A(k) + A(k - 1)B(n)A(n)B(k) + B(n)B(k - 1)] \]

\[ D = \begin{vmatrix} A(n + k) & B(n + k) \\ A(n) & B(n) \end{vmatrix} \]

\[ = A(n)A(k)B(n) + A(k - 1)B(n)B(k) - A(n)B(k) - A(n)B(k - 1). \]

The only possibilities for making \( D \) vanish are either \( n = k - 1 \) or \( n = k \).

When \( n = k - 1 \),

\[ D = A(k)A(k - 1)B(k - 1) - A(k - 1)^2B(k) = A(k - 1) \det(L) \neq 0. \]

When \( n = k \),

\[ D = A(k - 1)B(k - 1)^2 - A(k)B(k)B(k - 1) = -B(k) \det(L) \neq 0. \]

Hence \( x \) and \( y \) exist. Then let \( n = 0 \), we have

\[ A(2k) + xA(k) + yA(0) = 0, \quad B(2k) + xB(k) + yB(0) = 0. \]

Since \( A(0) = 1, B(0) = 0 \), we get

\[ x = -B(2k)/B(k), \quad y = A(k)B(2k)/B(k) - A(2k). \]

By Theorem 3, we obtain

\[ [A(2k)B(2k)] = [A(0)B(0)]L^2 = [1 0] L^2 = [A(k) + A(k - 1)B(k)B(k) + B(k)B(k - 1)] \]

Thus

\[ x = -B(2k)/B(k) = -(A(k) + B(k - 1)) = C(k) = A(k) + B(k - 1) \]

\[ y = A(k)A(k) + B(k - 1) - [A(k) + A(k - 1)B(k)] \]

\[ = A(k)B(k - 1) - A(k - 1)B(k) = \det(L) = (-1)^k. \]

Remark. Since \{ \( A(n) \) \} and \{ \( B(n) \) \} are also generated from \( P \), then

\[ A(n + 2k) - C(k)A(n + k) + (-1)^kA(n) = 0 \quad \text{and} \quad B(n + 2k) - C(k)B(n + k) + (-1)^kB(n) = 0. \]

By Theorem 3, this leads us to

\[ [A(n)B(n)]L^2 - C(k)L + (-1)^kI = 0 \rightarrow L^2 - C(k)L + \det(L)I = 0, \]

\( I \) is the identity matrix.

What happens when \( P = \{ \tilde{a} \} \) since \( k \) can be chosen as large as one desires?

**Theorem 5.** Suppose \{ \( r_n \) \} is generated from \( P = \{ \tilde{a} \} \) such that

\[ r_{n+2k} - C(k)r_{n+k} + (-1)^k r_n = 0. \]

Then \{ \( C(n) \) \} is also a sequence generated from \( P \) with \( C(0) = 2, C(-1) = -a. \)

**Proof.** Recall \( C(k) = A(k) + B(k - 1) \). Then

\[ C(k) - C(k - 1)a - C(k - 2) = \{ A(k) - A(k - 1)a - A(k - 2) \} - \{ B(k - 1) - B(k - 2)a - B(k - 3) \} = 0 \rightarrow C(k) = C(k - 1)a + C(k - 2). \]

Also,

\[ C(0) = A(0) + B(-1) = 2, \quad C(1) = A(1) + B(0) = a. \]

But then

\[ C(1) = C(0)a + C(-1) = C(-1) = -a. \]
Remark. Since \( \{C(n)\} \) is generated from \( P = \{\bar{a}\} \), there exists another sequence \( \{C'(n)\} \) such that
\[ C(n + 2k) - C(k)C(n + k) + (-1)^k C(n) = 0. \]

Notice that \( \{C'(n)\} = \{C(n)\} \). For example, when \( P = \{\bar{7}\} \), then
\[ \{A(n)\} = \{f_{n+1}\} \]
and \( \{B(n)\} = \{f_n\} \), \( C(n) = f_{n+1} + f_{n-1} \) for \( \{f_n\} \) is the Fibonacci sequence. Remember
\[ A(n + 2k) - C(k)A(n + k) + (-1)^k A(n) = 0 = f_{n+2k+1} - (f_{k+1} + f_{k-1})f_{n+k+1} + (-1)^k f_{n+1} = 0 \]
and
\[ B(n + 2k) - C(k)B(n + k) + (-1)^k B(n) = 0 = f_{n+2k} - (f_{k+1} + f_{k-1})f_{n+k} + (-1)^k f_n = 0. \]

Also from Theorem 5 and the last remark,
\[ C(n + 2k) - C(k)C(n + k) + (-1)^k C(n) = 0 = \{f_{n+2k+1} + f_{n+k-1}\} - (f_{k+1} + f_{k-1})\{f_{n+k+1} + f_{n+k-1}\} + (-1)^k\{f_{n+1} + f_{n-1}\} = 0. \]

**Theorem 6.** Suppose \( \{r_n\} \) is generated from \( P = \{\bar{a}\} \), then there exist \( x \) and \( y \) such that \( u > s > t > 0 \),
\[ r_{n+u} = x r_{n+s} + y r_{n+t} = 0. \]

\( x \) and \( y \) rational.

**Proof.** Think of \( n \) as \( k \) since the periodicity can vary.

Then follow the proof for Theorem 4. Carrying out the proof, we also find that
\[ x = \begin{bmatrix} A(u) & B(u) \\ A(t) & B(t) \end{bmatrix}, \quad y = \begin{bmatrix} A(s) & B(s) \\ A(t) & B(t) \end{bmatrix}. \]

In particular, when \( P = \{\bar{7}\} \), we get
\[ f_{n+u} = \begin{vmatrix} f_{u+1} & f_u \\ f_{u-1} & f_{u-2} \end{vmatrix} f_{n+2} - \begin{vmatrix} f_{u+1} & f_u \\ f_{u+1} & f_u \end{vmatrix} f_{n+1} = 0. \]

For example, when \( u = 9, s = 6 \) and \( t = 2 \),
\[ f_{n+9} - (13/3)f_{n+6} + (2/3)f_{n+2} = 0. \]

We are going to relate some of the above results to Continued Fractions.

A simple purely periodic continued fraction is denoted by \( c = [a_1, \ldots, a_k] \). If we take \( P = \{a_1, \ldots, a_k\} \), then immediately we see that \( A(n)/B(n) \) is the \( n^{th} \) convergent of \( c \). We also know that
\[ A(n + 2k) - C(k)A(n + k) + (-1)^k A(n) = 0 \quad \text{and} \quad B(n + 2k) - C(k)B(n + k) + (-1)^k B(n) = 0. \]

If we regard these as second-order difference equations, then the auxiliary quadratic equation for them is
\[ x^2 - C(k)x + (-1)^k = 0 \]
and
\[ x = \{C(k) \pm \sqrt{C(k)^2 - 4(-1)^k}\}/2, \quad C(k)^2 - 4(-1)^k > 0. \]

Let \( m_1, m_2 \) be the distinct zeros such that \( |m_1| > |m_2| \), then \( A(nk + u) = a_1 m_1^n + \beta_1 m_2^n \),
\[ B(nk + u) = a_2 m_1^n + \beta_2 m_2^n, \quad u < k. \]

By choosing the appropriate initial conditions for \( \{A(n)\} \) and \( \{B(n)\} \), respectively, we can solve for \( a_1, \beta_1 \) and \( a_2, \beta_2 \). One can take \( A(n), A(k + u) \) to be the initial conditions for \( \{A(n)\} \) and \( B(u), B(k + u) \) for \( \{B(n)\} \). Then the \((nk + u)\)th convergent of \( c \) is given by
\[ \frac{A(nk + u)}{B(nk + u)} = \frac{\alpha_1 + \beta_1(m_2/m_1)^n}{\alpha_2 + \beta_2(m_2/m_1)^n} \]

Hence limit of
\[ c = \lim_{n \to \infty} \left\{ \frac{A(nk + u)}{B(nk + u)} \right\} = \frac{\alpha_1}{\alpha_2}. \]

Notice that \( \alpha_1 \) and \( \alpha_2 \) are quadratic irrationals. Is the limit unique? Yes, by Theorem 3, we have
\[
\begin{vmatrix}
A(nk + u) & B(nk + u) \\
A(nk + v) & B(nk + v)
\end{vmatrix} = \det(L^n) \begin{vmatrix} A(u) & B(u) \\ A(v) & B(v) \end{vmatrix} = \pm 0,
\]
\( \sigma \) is a constant. Then
\[ \frac{A(nk + u)}{B(nk + u)} - \frac{A(nk + v)}{B(nk + v)} = \pm 0 \]
As \( n \to \infty \),
\[ \frac{A(nk + u)}{B(nk + u)} - \frac{A(nk + v)}{B(nk + v)} = 0. \]

If \( c = \{a_1, \ldots, a_j, a_{j+1}, \ldots, a_{j+k}\} \), then take
\[ P = \{a_1, \ldots, a_k, \ldots, a_{j+k}\} \]
as the generating sequence, the limit of \( c \) is then given by
\[ \lim_{n \to \infty} \frac{A(nk + u + j)}{B(nk + u + j)}, \quad u > 0. \]

**Remark.** Actually we have proved just now a theorem in continued fractions: A continued fraction \( c \) is periodic iff \( \sigma \) is a quadratic irrational, for which \( c \) is the continued fraction expansion.

**ADDITIVE PARTITIONS II**

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*Theorem (Hoggatt).* The Tribonacci Numbers,
\[ 1, 2, 4, 7, 13, 24, \ldots, T_{n+3} = T_{n+2} + T_{n+1} + T_n. \]
with 3 added to the set uniquely split the positive integers and each positive integer \( n \neq 3 \) or \( \neq T_m \) is the sum of two elements of \( A_0 \) or two elements of \( A_1 \). (See "Additive Partitions I," page 166.)

*Conjecture.* Let \( A \) split the positive integers into two sets \( A_0 \) and \( A_1 \) and be such that \( p \notin A_0 \cup \{1,2\} \), and \( p \) is representable as the sum of two elements of \( A_0 \) or the sum of two elements of \( A_1 \). We call such a set saturated (that is \( A \cup \{1,2\} \)). Krishnaswami Alladi asks: "Does a saturated set imply a unique additive partition?" My conjecture is that the set \( \{1, 2, 3, 4, 8, 13, 24, \ldots\} \) is saturated but does not cause a unique split of the positive integers. Here we have added 3 and 8 to the Tribonacci sequence and deleted the 7. Paul Bruckman points out that this fails for 41. **EDITOR**

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