ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-274 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

It has been shown (*The Fibonacci Quarterly,* Vol. 2, No. 2 (April, 1964), pp. 261–266) that if

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{then} \quad Q^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1} - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} .$$

Generalize the matrix Q to solutions of the difference equation

$$U_n = rU_{n-1} + sU_{n-2} ,$$

where r and s are arbitrary real numbers, $U_0 = 0$ and $U_1 = 1$.

H-275 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let P_n denote the Pell Sequence defined as follows: $P_1 = 1, P_2 = 2, P_{n+2} = 2P_{n+1} + P_n \ (n \geq 1)$. Consider the array below.

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1    2    5    12     29     70 ···    (P_n)
   1    3    7     17    41    ···
      2    4    10     24    ···
         2    6     14    ···
            4    8    ···
               4  ···
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Each row is obtained by taking differences in the row above.

Let D_n denote the left diagonal sequence in this array; i.e.,

$$D_1 = D_2 = 1, \quad D_3 = D_4 = 2, \quad D_5 = D_6 = 4, \quad D_7 = D_8 = 8, \cdots .$$

(i) Show $$D_{2n-1} = D_{2n} = 2^{n-1} \quad (n \geq 1).$$

(ii) Show that if $F(x)$ represents the generating function for $\{P_n\}_{n=1}^{\infty}$ and $D(x)$ represents the generating function for $\{D_n\}_{n=1}^{\infty}$, then

$$D(x) = \frac{1}{1+x} F\left(\frac{x}{1+x}\right) .$$

SOLUTIONS

DOUBLE YOUR FUN

H-255 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\binom{2m+2n}{j+k}\binom{2m+2n}{2m-j+k} = (-1)^{m+n}\frac{(3m+3n)!\,(2m)!\,(2n)!}{m!\,n!\,(m+n)!\,(2m+n)!(m+2n)!} .$$

Solution by the Proposer.

We shall use the following Saalschützian theorem for double series:

$$(1.1)\qquad \sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(a)_{r+s}(b)_r(c)_s}{r!\,s!\,(c)_{r+s}(d)_r(d')_s} = (-1)^{m+n}\frac{(c-a)_{m+n}(c-a-b')_m(c-a-b)_n}{(c)_{m+n}(c-a-b)_m(c-a-b')_n} ,$$

where

$$a+1 = d+d',\quad c+d = a+b-m+1,\quad c+d' = a+b'-n+1 .$$

(For proof of (1) see *Journal London Math. Soc.*, 38 (1968), pp. 415–418.)

In (1) replace b, b' by $b+m$, $b'+n$, respectively; also replace m,n by j,k. Then (1) becomes

$$\sum_{r=0}^{j}\sum_{s=0}^{k}\frac{(-j)_r(-k)_s(a)_{r+s}(b+j)_r(b'+k)_s}{r!\,s!\,(c)_{r+s}(d)_r(d')_s} = \frac{(c-a)_{j+k}(-d'-k+1)_j(-d-j+1)_k}{(c)_{j+k}(d)_j(d')_k} ,$$

where now

$$(2)\qquad a+1 = d+d',\quad c = b+d' = b'+d .$$

Then

$$\sum_{j,k=0}^{\infty}\frac{(b)_j(b')_k}{j!\,k!}\,\frac{(c-a)_{j+k}(-d'-k+1)_j(-d-j+1)_k}{(c)_{j+k}(d)_j(d')_k}x^jy^k$$

$$= \sum_{j,k=0}^{\infty}\frac{(b)_j(b')_k}{j!\,k!}x^jy^k\sum_{r=0}^{j}\sum_{s=0}^{k}\frac{(-j)_r(-k)_s(a)_{r+s}(b+j)_r(b'+k)_s}{r!\,s!\,(c)_{r+s}(d)_j(d')_k}$$

$$= \sum_{r,s=0}^{\infty}(-1)^{r+s}\frac{(a)_{r+s}(b)_{2r}(b')_{2s}}{r!\,s!\,(c)_{r+s}(d)_r(d')_s}x^ry^s\sum_{j,k=0}^{\infty}\frac{(b+2r)_j(b+2s)_k}{j!\,k!}x^jy^k$$

$$= \sum_{r,s=0}^{\infty}(-1)^{r+s}\frac{(a)_{r+s}(b)_{2r}(b')_{2s}}{r!\,s!\,(c)_{r+s}(d)_r(d')_s}x^ry^r(1-x)^{-b-2r}(1-y)^{-b'-2s} ,$$

where a, b, b', c, d, d' satisfy (2).

Now take $b=-2m$, $c=-2n$. Then

$$d = c+2n,\qquad d' = c+2m,\qquad a+1 = 2c+2m+2n .$$

The above identity becomes

$$(3)\qquad \sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\frac{(-c-2m-2n+1)_{j+k}(-c-2m-k+1)_j(-c-2n-j+1)_k}{(c)_{j+k}(c+2n)_j(c+2m)_k}x^jy^k$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\frac{(2c+2m+2n-1)_{r+s}(-2m)_{2r}(-2n)_{2s}}{r!\,s!\,(c)_{r+s}(c+2n)_r(c+2m)_s}x^ry^s(1-x)^{2m-2r}(1-y)^{2m-2s} .$$

We now take $x=y=1$, $c=p+1$, where p is a non-negative integer. Then (3) reduces to

$$(4)\qquad \sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\binom{2m+2n+2p}{j+k+p}\binom{2m+2n+2p}{2m+p-j+k}$$

$$= (-1)^{m+n}\frac{(2m)!\,(2n)!\,(3m+3n+2p)!\,(2m+2n+2p)!}{m!\,n!\,(m+n+p)!\,(2m+2n+p)!\,\,(2m+n+p)!\,(m+2n+p)!} .$$

For $p = 0$, (4) gives the stated result.

STAGGERING SUM

H-257 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Consider the array, D, indicated below in which F_{2n+1} $(n = 0, 1, 2, \cdots)$ is written in staggered columns.

$$D:\quad \begin{array}{rrrrrr} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 2 & 1 & & & \\ 13 & 5 & 2 & 1 & & \\ 34 & 13 & 5 & 2 & 1 & \\ 89 & 34 & 13 & 5 & 2 & 1 \end{array}$$

(i) Show that the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.

(ii) Show that the rising diagonal sums are $F_{n+1}F_{n+2}$ $(n = 0, 1, 2, \cdots)$.

(iii) Show that if the columns are multiplied by 1, 2, 3, ⋯ sequentially to the right, then the row sums are $F_{2n+3} - 1$ $(n = 0, 1, 2, \cdots)$.

Solution by George Brezsenyi, Lamar University, Beaumont, Texas.

(i) The sum of the entries of the n^{th} row is easily seen to be

$$\sum_{k=0}^{n} F_{2k+1},$$

which is well known to be F_{2n+2}.

(ii) The sums seemingly depend upon the parity of n. If n is odd, say $n = 2m + 1$, then the rising diagonal sum is

$$\sum_{k=0}^{m} F_{4k+3},$$

which may be shown to equal $F_{2m+2}F_{2m+3}$, or $F_{n+1}F_{n+2}$, by mathematical induction. Similarly, if n is even, say $n = 2m$, then the desired sum

$$\sum_{k=0}^{m} F_{4k+1}$$

yields upon evaluation $F_{2m+1}F_{2m+2}$, which is also equal to $F_{n+1}F_{n+2}$.

(iii) To resolve this part of the problem we show that

$$\sum_{k=0}^{n} (n+1-k)F_{2k+1} = F_{2n+3} - 1.$$

In $n = 0$, the result is trivial. Assume it for $n = m$. Then for $n = m + 1$ we have

$$\sum_{k=0}^{m+1} ((m+1)+1-k)F_{2k+1} = \sum_{k=0}^{m} (m+2-k)F_{2k+1} + F_{2m+3}$$

$$= \sum_{k=0}^{m} (m+1-k)F_{2k+1} + \sum_{k=0}^{m} F_{2k+1} + F_{2m+3}$$

$$= F_{2m+3} - 1 + F_{2m+2} + F_{2m+3} = F_{2(m+1)+3} - 1.$$

Thus the result holds for $n = m + 1$. This completes the induction.

Also solved by W. Brady, A. Shannon, G. Lord, P. Bruckman, F. Higgins and the Proposer.

THE SIGMA STRAIN

H-258 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S = \sum x^a y^b z^c t^d,$$

where the summation is over all non-negative a, b, c, d, such that

$$\begin{cases} 2a \leqslant b+c+d \\ 2b \leqslant a+c+d \\ 2c \leqslant a+b+d \\ 2d \leqslant a+b+c \end{cases}$$

Solution by the Proposer.

Let

$$\begin{cases} a' = -2a+b+c+d \\ b' = a-2b+c+d \\ c' = a+b-2c+d \\ d' = a+b+c-2d. \end{cases}$$

Then a', b', c', d' are non-negative and

$$\begin{cases} 3a = b'+c'+d' \\ 3b = a'+c'+d' \\ 3c = a'+b'+d' \\ 3d = a'+b'+c' \end{cases} .$$

Thus

$$\begin{cases} b'+c'+d' \equiv 0 \\ a'+c'+d' \equiv 0 \\ a'+b'+d' \equiv 0 \\ a'+b'+c' \equiv 0 \end{cases} \pmod 3.$$

This implies

$$a' \equiv b' \equiv c' \equiv d' \pmod 3$$

and conversely.

Hence

$$S = S_0 + S_1 + S_2,$$

where

$$S_i = \sum_{\substack{a',b',c',d'=0 \\ a'\equiv b'\equiv c'\equiv d'\equiv i \pmod 3}}^{\infty} x^{\frac{1}{3}(b'+c'+d')} y^{\frac{1}{3}(a'+c'+d')} z^{\frac{1}{3}(a'+b'+d')} t^{\frac{1}{3}(a',b',c')} \qquad (i = 0, 1, 2).$$

Put $a' = 3a + i$, etc. Then

$$S_i = (xyzt)^i \sum_{a,b,c,d=0}^{\infty} x^{b+c+d} y^{a+c+d} z^{a+b+d} t^{a+b+c} = \frac{(xyzt)^i}{(1-yzt)(1-xzt)(1-xyt)(1-xyz)} \qquad (i = 0,1,2).$$

so that

$$S = \frac{1 + xyzt + (xyzt)^2}{(1-yzt)(1-xzt)(1-xyt)(1-xyz)} .$$

POSITIVELY!

H-259 Proposed by R. Finkelstein, Tempe, Arizona.

Let p be an odd prime and m an odd integer such that $m \not\equiv 0 \pmod p$. Let $F_{mp} = F_p \cdot Q$. Can $(F_p, Q) > 1$?
[Continued on page 288.]