GAUSSIAN FIBONACCI NUMBERS

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The purpose of this note is to present a natural manner of extension of the Fibonacci numbers into the complex plane. The extension is analogous to the analytic continuation of solutions of differential equations. Although, in general, it does not guarantee permanence of form, in case of the Fibonacci numbers even that requirement is satisfied. The resulting complex Fibonacci numbers are, in fact, Gaussian integers. The applicability of this generalization will be demonstrated by the derivation of two interesting identities for the classical Fibonacci numbers.

The notion of monodiffricity was introduced by Rufus P. Isaacs [1, 2] in 1941; for references to the more recent literature the reader is directed to two papers by the present author [3, 4]. The domain of definition of monodiffric functions is the set of Gaussian integers; a complex-valued function f is said to be monodiffric at $z = x + yi$ if

$$\frac{1}{i}\,[f(z+i) - f(z)] = f(z+1) - f(z). \tag{1}$$

As Isaacs already observed, if f is defined on the set of integers, then the requirement of monodiffricity determines f uniquely at the Gaussian integers of the upper half-plane. We term this extension monodiffric continuation. Kurowsky [5] showed that the functional values of f may be calculated by use of the formula

$$f(x+yi) = \sum_{k=0}^{y} \binom{y}{k} i^k \Delta^k f(x), \tag{2}$$

where the operator Δ^k is defined by the relations

$$\Delta^0 f(x) = f(x), \quad \Delta^1 f(x) = f(x+1) - f(x) \text{ and } \Delta^k f(x) = \Delta^{k-1}(\Delta^1 f(x)) \text{ for } k \geqslant 2.$$

When applied to the Fibonacci numbers Δ^k behaves especially nicely; one may easily prove that

$$\Delta^k F_n = F_{n-k}\ .$$

Therefore, via Eq. (2), one may define the Gaussian Fibonacci numbers, F_{n+mi}, for n an integer, m a nonnegative integer by

$$F_{n+mi} = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k}\ . \tag{3}$$

The first few values of F_{n+mi} are tabulated below:

$-6+8i$	$3-4i$	$-3+4i$	0	$-3+4i$	$-3+4i$	$-6+8i$	$-9+12i$	$-15+20i$
-5	$3+i$	$-2+i$	$1+2i$	$-1+3i$	$5i$	$-1+8i$	$-1+13i$	$-2+21i$
$-1-2i$	$1+2i$	0	$1+2i$	$1+2i$	$2+4i$	$3+6i$	$5+10i$	$8+16i$
$1-i$	i	1	$1+i$	$2+i$	$3+2i$	$5+3i$	$8+5i$	$13+8i$
1	0	1	1	2	3	5	8	13

Figure 1

On the basis of Eq. (3) it is easily shown that

(4) $$F_{n+mi} = F_{(n-1)+mi} + F_{(n-2)+mi} ,$$

that is, for each fixed m, the sequences $\{Re(F_{n+mi})\}$ and $\{Im(F_{n+mi})\}$ are generalized Fibonacci sequences in the sense of Horadam [6].

Our first aim will be to utilize Eq. (4) in order to find a closed form for the Gaussian Fibonacci numbers. The development hinges upon the observation (easily proven by induction via Eq. (1)) that for each $m = 0, 1, 2, \cdots$,

$$F_{m+2mi} = 0,$$

and, consequently, with the help of Eq. (4), one can prove that

(5) $$F_{n+2mi} = F_{m+1+2mi} F_{n-m}$$

for each $n = 0, \pm 1, \pm 2, \cdots, m = 0, 1, 2, \cdots$.

Although one could show directly that

(6) $$F_{m+1+2mi} = (1+2i)^m ,$$

we shall provide a more insightful derivation. It is well known that if

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{then} \quad Q^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$$

for each $k = 0, \pm 1, \pm 2, \cdots$. Since a matrix must satisfy its characteristic equation, one may then write

$$Q^2 = Q + I.$$

With the help of this one finds that

$$(Q + iI)^2 = Q^2 + 2iQ - I = (1+2i)Q,$$

or, more generally, for $m = 0, 1, 2, \cdots$

$$(Q + iI)^{2m} = (1+2i)^m Q^m .$$

Expansion of the left member of this identity and multiplication by Q^{n-2m} yields

$$\sum_{k=0}^{2m} \binom{2m}{k} i^k Q^{n-k} = (1+2i)^m Q^{n-m} .$$

Finally, equating the first row second column entries of the two members of this matrix identity gives

(7) $$\sum_{k=0}^{2m} \binom{2m}{k} i^k F_{n-k} = (1+2i)^m F_{n-m} .$$

Since, in view of Eq. (3), the left members of Eqs. (5) and (7) are identical, Eq. (6) is proven.

The evaluation of the right member of Eq. (3) for odd m is easily accomplished now with the help of Eq. (1). The results may be summarized as follows:

(8a) $$F_{n+2mi} = (1+2i)^m F_{n-m}$$

(8b) $$F_{n+(2m+1)i} = (1+2i)^m [F_{n-m} + iF_{n-1-m}] .$$

It may be observed that for fixed odd positive integers, m, the sequences $\{F_{n+mi}\}$ are closely related to the generalized complex Fibonacci sequences studied by Horadam [7] and possess similar properties. One may also observe that Eq. (6) is a special case of Eq. (8a), arising when $n = m + 1$.

The identities,

(9a) $$\sum_{k=0}^{m} \binom{2m}{2k} (-1)^k F_{n-2k} = a_m F_{n-m}$$

(9b) $$\sum_{k=0}^{m} \binom{2m+2}{2k+1} (-1)^k F_{n-2k} = b_{m+1} F_{n-m} ,$$

promised earlier in the paper, are obtained by equating the real and the imaginary parts of Eq. (7). The numbers a_k and b_k, defined by

$$(1+2i)^k = a_k + b_k i,$$

may also be obtained with the help of the following algorithm (which is more in the spirit of the present publication): $a_0 = 1$, $b_0 = 0$ and for $k \geqslant 1$,

$$a_k = a_{k-1} - 2b_{k-1} \quad \text{and} \quad b_k = b_{k-1} + 2a_{k-1}.$$

The table below lists the first few values of a_k and b_k obtained in this manner:

n	a_n	b_n	n	a_n	b_n
1	1	2	11	6,469	−2,642
2	−3	4	12	11,753	10,296
3	−11	−2	13	−8,839	33,802
4	−7	−24	14	−76,443	16,124
5	41	−38	15	−108,691	−136,762
6	117	44	16	164,833	−354,144
7	29	278	17	873,121	−24,478
8	−527	336	18	922,077	1,721,764
9	−1,199	−718	19	−2,521,451	3,565,918
10	237	−3,116	20	−9,653,287	−1,476,984

Figure 2

To illustrate the results, we list below the evaluation of Eqs. (9a) and (9b) for $m = 5$:

$$F_n - 45F_{n-2} + 210F_{n-4} - 210F_{n-6} + 45F_{n-8} - F_{n-10} = 41F_{n-5},$$

$$12F_n - 220F_{n-2} + 792F_{n-4} - 792F_{n-6} + 220F_{n-8} - 12F_{n-10} = 44F_{n-5},$$

which, upon simplification, may be combined into the following elegant relationship:

(11) $$F_n - 5F_{n+2} - 9F_{n+5} + 5F_{n+8} - F_{n+10} = 0.$$

Other simple identities arising as special cases include:

(12) $$F_n - 3F_{n+2} + F_{n+4} = 0,$$

(13) $$F_n + 4F_{n+3} - F_{n+6} = 0,$$

and

(14) $$F_n - 12F_{n+2} + 29F_{n+4} - 12F_{n+6} + F_{n+8} = 0.$$

In conclusion we note that the entire development can be extended to the study of generalized Fibonacci numbers. In fact, if the sequence H_n is defined by

$$H_0 = p, \quad H_1 = q, \quad H_n = H_{n-1} + H_{n-2} \quad \text{for} \quad n \geqslant 2,$$

where p and q are arbitrary integers, then Eqs. (9a) and (9b) will readily generalize to

(15a) $$\sum_{k=0}^{m} \binom{2m}{2k} (-1)^k H_{n-2k} = a_m H_{n-m}$$

and

(15b) $$\sum_{k=0}^{m} \binom{2m+2}{2k+1} (-1)^k H_{n-2k} = b_{m+1} H_{n-m},$$

respectively.

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CONSTANTLY MEAN

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The golden mean is quite absurd;
It's not your ordinary surd.
If you invert it (this is fun!),
You'll get itself, reduced by one;
But if increased by unity,
This yields its square, take it from me.

Alone among the numbers real,
It represents the Greek ideal.
Rectangles golden which are seen,
Are shaped such that this golden mean,
As ratio of the base and height,
Gives greatest visual delight.

Expressed as a continued fraction,
It's one, one, one, ···, until distraction;
In short, the simplest of such kind
(Doesn't this really blow your mind?)
And the convergents, if you watch,
Display the series Fibonacc'
In both their bottom and their top,
That is, until you care to stop.

Since it belongs to F-root-five
Its value's tedious to derive.
These properties are quite unique
And make it something of a freak.
Yes, one-point-six-one-eight-oh-three,
You're too irrational for me.