ON MINIMAL NUMBER OF TERMS IN REPRESENTATION OF NATURAL NUMBERS AS A SUM OF FIBONACCI NUMBERS

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Let $f(k)$ denote this number for any natural number k. It is shown that $f(k) \leqslant n$ for $k < F_{2n+2} - 2$, $f(k) = n$ for $k = F_{2n+2} - 2$ and $f(k) = n + 1$ for $k = F_{2n+2} - 1$.

1. A *base* for natural numbers is any sequence S of positive integers for which numbers n and N may be found such that any positive integer $\geqslant N$ may be represented as a sum of $\leqslant n$ members of S. Any arithmetical progression

(1) $$1,\ 1+d,\ 1+2d,\ \cdots,$$

where d is an integer and $d > 1$, is a base (it is enough to take $n = d$, $N = 1$). A geometrical progression

(2) $$1,\ q,\ q^2,\ \cdots,$$

where q is an integer and $q > 1$, is not a base; if we take for any positive integers n and N the number

$$\sum_{i=0}^{m} q^i = \frac{q^{m+1} - 1}{q - 1},$$

where

$$m = \max\,(n, [lg_q\{1 + N(q-1)\}]),$$

is greater than N, but may not be represented as a sum of $\leqslant n$ numbers of progression (2). The sequence of the Fibonacci numbers is defined as $F_i = i$, where $i = 1,2$; $F_i = F_{i-1} + F_{i-2}$, where $i > 2$. This sequence may be considered additive by definition, but it increases faster than any arithmetical progression of type (I). On the other hand a specific characteristic of Fibonacci numbers

$$\lim_{i\to\infty} \frac{F_{i+1}}{F_i} = \frac{\sqrt{5}+1}{2}$$

shows that they increase asymptotically as a geometrical progression with a denominator

$$\frac{\sqrt{5}+1}{2} = q^*;$$

however, $q^* < 2$, i.e., Fibonacci numbers increase more slowly than any geometrical progression of type (2). We show that Fibonacci numbers, in the representation of the positive integers as a sum of these numbers, act as a geometrical progression of type (2). Let us call

$$k = \sum_{i=1}^{f} F_{m_i}, \qquad m_i \leqslant m_{i-1},$$

a *correct* decomposition, if $f = 1$, or if $f > 1$ we have $m_i < m_{i-1} - 1$ for all $i \in [2,f]$.

The theorem of Zeckendorf gives that for any positive integer there exists a correct decomposition; moreover any decomposition of the positive integer into a sum of Fibonacci numbers contains no fewer terms than its correct decomposition.

2. *Theorem 1.*

(1) For any positive integer n the number $F_{2n+2} - 1$ is the smallest number which is not representable as a sum of $\leqslant n$ Fibonacci numbers.

(2) Number $F_{2n+2} - 1$ may be represented as a sum of $n + 1$ Fibonacci numbers.

(3) Number $F_{2n+2} - 2$ is not representable as a sum of $< n$ Fibonacci numbers.

Indeed, if $n = 1$, theorem is evident. Let us assume that the theorem is correct for $n \leqslant m$. The numbers of segment $[1, F_{2m+2} - 2]$ may be represented for part (1) of the theorem, as a sum of $\leqslant m$ Fibonacci numbers. Number $(F_{2m+2} - 2) + 1 = F_{2m+2} - 1$ may be represented for part (2) as a sum of $m + 1$ Fibonacci numbers. Number $(F_{2m+2} - 2) + 2 = F_{2m+2}$ is a Fibonacci number. The numbers of segment

(3) $$[F_{2m+2} + 1, \quad F_{2m+2} + (F_{2m+1} - 1)]$$

are sums of number F_{2m+2} and of the corresponding numbers of segment $[1, F_{2m+1} - 1]$, which for part (1) of the theorem (since $F_{2m+1} - 1 \leqslant F_{2m+2} - 2$) are representable as a sum of $\leqslant m$ Fibonacci numbers. Number $F_{2m+2} + (F_{2m+1} - 1) + 1 = F_{2m+3}$ is a Fibonacci number. The numbers of the segment

$$[F_{2m+3} + 1, \quad F_{2m+3} + (F_{2m+2} - 2)]$$

are representable as a sum of $\leqslant m + 1$ Fibonacci numbers for the same reason as for the numbers of segment (3); though in this case we have the number F_{2m+3} and not F_{2m+2}. Thus all numbers not greater than

$$F_{2m+3} + (F_{2m+2} - 2) = F_{2(m+1)+2} - 2$$

are representable as sums of $\leqslant m + 1$ Fibonacci numbers. A correct decomposition of numbers $F_{2m+2} - 2$ and $F_{2m+2} - 1$ contains respectively (on the basis of the inductive assumptions) m and $m + 1$ terms. If to these decompositions we add on the left-hand side the term F_{2m+3} we obtain the correct decomposition of numbers $F_{2m+4} - 2$ and $F_{2m+4} - 1$. These latter contain respectively $m + 1$ and $m + 2$ terms. From this and from the theorem of Zeckendorf it follows that numbers $F_{2(m+1)+2} - 2$ and $F_{2(m+1)+2} - 1$ may be represented respectively as the sums of $m + 1$ (but not less) and respectively $m + 2$ (but not less) Fibonacci numbers.

By the way, it is clear that

$$F_{2n+2} - 2 = \sum_{i=1}^{2n} F_i = \sum_{i=1}^{n} F_{2i+1} .$$

One of more detailed works on these problems is [2].

REFERENCES

1. E. Zeckendorf, "Representation des nombres naturels par une somme de nombres de Fibonacci ou des nombres de Lucas," *Bull. Soc. Royale Sci. de Liege,* 3–9, 1972, pp. 779–182.
2. L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville, "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Special Issue, January 1972), pp. 1–28.

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LETTER TO THE EDITOR

April 28, 1970

In regard to the two articles, "A Shorter Proof" by Irving Adler (December, 1969 *Fibonacci Quarterly*) and "1967 as the Sum of Three Squares," by Brother Alfred Brousseau (April, 1967 *Fibonacci Quarterly*), the general result is as follows:

$$x^2 + y^2 + z^2 = n$$

is solvable if and only if n is not of the form $4^t(8k + 7)$, for $t = 0, 1, 2, \cdots, k = 0, 1, 2, \cdots$. See [1].

Since 1967 = 8(245) + 7, $1967 \neq x^2 + y^2 + z^2$. A lesser result known to Fermat and proven by Descartes is that no integer $8k + 7$ is the sum of three rational squares [2]. The *really* short and usual proof is:

For x, y, and z any integers, $x^2 \equiv 0, 1$, or 4 (mod 8) so that $x^2 + y^2 + z^2 \equiv 0, 1, 2, 3, 4, 5$, or 6 (mod 8) or $x^2 + y^2 + z^2 \not\equiv 7$ (mod 8).

REFERENCES

1. William H. Leveque, *Topics in Number Theory,* Vol. I, p. 133.
2. Leonard E. Dickson, *History of the Theory of Numbers,* Vol. II, Chap. VII, p. 259.

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