GENERATING FUNCTIONS FOR POWERS OF CERTAIN SECOND-ORDER RECURRENCE SEQUENCES

BLAGOJ S. POPOV
Institut de Mathematiques, Skoplje, Jugoslavia

1. INTRODUCTION

Let $u(n)$ and $v(n)$ be two sequences of numbers defined by

(1) $$u(n) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}, \quad n = 0, 1, 2, \cdots$$

and

(2) $$v(n) = r_1^n + r_2^n, \quad n = 0, 1, 2, \cdots,$$

where r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$.

It is known that the generating functions of these sequences are

$$u_1(x) = \left(1 + \frac{b}{a}x + \frac{c}{a}x^2\right)^{-1} \quad \text{and} \quad v_1(x) = \left(2 + \frac{b}{a}x\right)\left(1 + \frac{b}{a}x + \frac{c}{a}x^2\right)^{-1}.$$

We put

(3) $$u_k(x) = \sum_{n=0}^{\infty} u^k(n)x^n$$

and

(4) $$v_k(x) = \sum_{n=0}^{\infty} v^k(n)x^n.$$

J. Riordan [1] found a recurrence for $u_k(x)$ in the case $b = c = -a$. L. Carlitz [2] generalized the result of Riordan giving the recurrence relations for $u_k(x)$ and $v_k(x)$. A. Horadam [3] obtained a recurrence which unifies the preceding ones. He and A. G. Shannon [4] considered third-order recurrence sequences, too.

The object of this paper is to give the new recurrence relations for $u_k(x)$ and $v_k(x)$ such as the explicit form of the same generating functions. The generating functions of $u(n)$ and $v(n)$ for the multiple argument will be given, too. We use the result of E. Lucas [5].

2. RELATIONS OF $u(n)$ AND $v(n)$

From (1) and (2) we have

$$4r_i^{m+n+2} = \Delta u(n)u(m) + v(n+1)v(m+1) + (-1)^{i-1}\sqrt{\Delta}\,(u(n)v(m+1) + u(m)v(n+1)), \quad i = 1, 2,$$

with $\Delta = (b^2 - 4ac)/a^2$.

Then it follows that

$$2u(m+n+1) = u(n)v(m+1) + u(m)v(n+1)$$

$$2v(m+n+2) = v(n+1)v(m+1) + \Delta u(n)u(m).$$

Since

$$u(-n-1) = -q^{-n}u(n-1), \quad v(-n) = -q^{-n}v(n),$$

we find the relations

(5) $$u((n+2)m - 1) = u((n+1)m - 1)v(m) - q^m u(nm - 1),$$

(6) $$v(nm) = v((n-1)m)v(m) - q^m v((n-2)m).$$

From the identity

$$r_1^{kn} + r_2^{kn} = \sum_{r=0}^{[k/2]} (-1)^r \frac{k}{k-r} C_{k-r}^r (r_1^n + r_2^n)^{k-2r} (r_1 r_2)^{rn},$$

if we put $u(n)$ and $v(n)$ we get

(7) $$v(kn) = \sum_{r=0}^{[k/2]} (-1)^r \frac{k}{k-r} C_{k-r}^r q^{rn} v^{k-2r}(n), \quad k \geq 1.$$

Similarly, from

$$2r_i^{n+1} = v(n+1) + (-1)^{i-1}\sqrt{\Delta}\, u(n), \quad i = 1, 2,$$

and taking into consideration

$$\sum_{s=0} \binom{p+s}{s} \binom{2p+m}{2p+2s} = 2^{m-1} \frac{2p+m}{m} \binom{m+p-1}{p},$$

we obtain

(8) $$\sum_{r=0}^{[k/2]} \Delta^{[k/2]-r} \frac{k}{k-r} C_{k-r}^r q^{r(n+1)} u^{k-2r}(n) = \lambda_k(n),$$

where

$$\lambda_k(n) = \begin{cases} u(k(n+1)-1), & k \text{ odd}, \\ v(k(n+1)), & k \text{ even}. \end{cases}$$

3. GENERATING FUNCTIONS OF $u(n)$ AND $v(n)$ FOR MULTIPLE ARGUMENT

The relations (5) and (6) give us the possibility to find the generating functions of $u(n)$ and $v(n)$ when the argument is a multiple. Indeed, we obtain from (5)

(9) $$(1 - v(m)x + q^m x^2) u(m,x) = u(m-1),$$

where

(10) $$u(m,x) = \sum_{n=0}^{\infty} u((n+1)m - 1)x^n.$$

From (6) we have

(11) $$(1 - v(m)x + q^m x^2) v(m,x) = v(m) - q^m v(0)x,$$

where

(12) $$v(m,x) = \sum_{n=0}^{\infty} v((n+1)m)x^n.$$

We find also

(13) $$(1 - v(m)x + q^m x^2) \tilde{v}(m,x) = v(0) - v(m)x,$$

with

$$\tilde{v}(m,x) = v(0) + v(m,x)x.$$

4. RECURRENCE RELATIONS OF $u_k(x)$ AND $v_k(x)$

Let us now return to (8) and consider the sum

$$\sum_{r=0}^{[k/2]} \Delta^{[k/2]-r} \frac{k}{k-r} C_{k-r}^r q^r \sum_{n=0}^{\infty} u^{k-2r}(n)(q^r x)^n = \sum_{n=0}^{\infty} \lambda_k(n) x^n$$

which by (3), (10) and (12) yields the following relation

$$\Delta^{[k/2]} u_k(x) = \lambda(k,x) - \sum_{r=1}^{[k/2]} \Delta^{[k/2]-r} \frac{k}{k-r} C_{k-r}^r q^r u_{k-2r}(q^r x),$$

where

$$\lambda(k,x) = \begin{cases} u(k,x), & k \text{ odd} \\ v(k,x), & k \text{ even}. \end{cases}$$

Similarly from (7) for $v_k(x)$ follows

$$v_k(x) = \tilde{v}(k,x) + \sum_{r=1}^{[k/2]} (-1)^{r-1} \frac{k}{k-r} C_{k-r}^r v_{k-2r}(q^r x).$$

5. EXPLICIT FORM OF $u_k(x)$ AND $v_k(x)$

Next we construct the powers for $u(n)$ and $v(n)$. From (1) and (2) we obtain

(14) $$\Delta^{[k/2]} u^k(n) = \sum_{r=0}^{[k/2]} (-1)^r C_k^r q^{r(n+1)} \lambda_{k-2r}(n),$$

and

(15) $$v^k(n) = \sum_{r=0}^{[k/2]} C_k^r q^{rn} \tilde{v}((k-2r)n),$$

where

$$\tilde{v}(t) = \begin{cases} v(t), & t \neq 0, \\ \tfrac{1}{2}v(t), & t = 0. \end{cases}$$

Hence we multiply each member of the equations (14) and (15) by x^n and sum from $n = 0$ to $n = \infty$. By (3) and (4) the following generating functions for powers of $u(n)$ and $v(n)$ are obtained:

$$\Delta^{[k/2]} u_k(x) = \sum_{r=0}^{[k/2]} (-1)^r C_k^r q^r \lambda(k-2r, q^r x),$$

and

$$v_k(x) = \sum_{r=0}^{[k/2]} C_k^r v(k-2r, q^r x).$$

If we replace $u(m,x)$, $v(m,x)$ and $\tilde{v}(m,x)$ from (9), (11) and (13), we get

$$\Delta^{[k/2]} u_k(x) = \sum_{r=0}^{[k/2]} \frac{(-1)^r C_k^r q^r \mu_{kr}(x)}{1 - v(k-2r)q^r x + q^k x^2},$$

where

$$\mu_{kr} = \begin{cases} u(k-2r-1), & k \text{ odd}, \\ v(k-2r) - q^r v(0)x, & k \text{ even}, k \neq 2r \\ \tilde{v}(k-2r) - q^r v(0)x, & k = 2r, \end{cases}$$

and

$$v_k(x) = \sum_{r=0}^{[k/2]} \frac{C_k^r \omega_{kr}(x)}{1 - v(k-2r)q^r x + q^k x^2},$$

where

$$\omega_{kr} = \begin{matrix} v(0) - q^r v(k-2r)x, & k \neq 2r, \\ \tilde{v}(0) - q^r \tilde{v}(k-2r)x, & k = 2r. \end{matrix}$$

REFERENCES

1. J. Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math J.*, V. 29 (1962), 5–12.
2. L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," *Duke Math J.*, Vol. 29 (1962), pp. 521–537.
3. A.F. Horadam, "Generating Functions for Powers of Certain Generalized Sequences of Numbers," *Duke*

Math. J., Vol. 32 (1965), pp. 437–446.
4. A.G. Shannon and A.F. Horadam, "Generating Functions for Powers of Third-Order Recurrence Sequences," *Duke Math. J.,* Vol. 38 (1971), pp. 791–794.
5. E. Lucas, *Theorie des Nombres,* Paris, 1891.

****

A SET OF GENERALIZED FIBONACCI SEQUENCES SUCH THAT EACH NATURAL NUMBER BELONGS TO EXACTLY ONE

KENNETH B. STOLARSKY
University of Illinois, Urbana, Illinois 61801

1. INTRODUCTION

We shall prove there is an infinite array

1	2	3	5	8	.	.	.
4	6	10	16	26	.	.	.
7	11	18	29	47	.	.	.
9	15	24	39	63	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

in which every natural number occurs exactly once, such that past the second column every number in a given row is the sum of the two previous numbers in that row.

2. PROOF

Let α be the largest root of $z^2 - z - 1 = 0$, so $\alpha = (1+\sqrt{5})/2$. For every positive integer x let $f(x) = [\alpha x + \frac{1}{2}]$ where $[u]$ denotes the greatest integer in u. We require two lemmas: the first asserts that $f(x)$ is one-to-one, and the second asserts that the iterates of $f(x)$ form a sequence with the Fibonacci property.

Lemma 1. If x and y are positive integers and $x > y$ then $f(x) > f(y)$.

Proof. Since $\alpha(x-y) > 1$ we have $(\alpha x + \frac{1}{2}) - (\alpha y + \frac{1}{2}) > 1$, so $f(x) > f(y)$.

Lemma 2. If x and y are integers, and $y = [\alpha x + \frac{1}{2}]$, then $x + y = [\alpha y + \frac{1}{2}]$.

Proof. Write $\alpha x + \frac{1}{2} = y + r$, where $0 < r < 1$. Then

$$(1+\alpha)x + \frac{\alpha}{2} = \alpha y + \alpha r$$

so

$$x + y + r - \frac{1}{2} + \frac{\alpha}{2} = \alpha y + \alpha r \quad \text{and} \quad \alpha y + \frac{1}{2} = x + y + \frac{\alpha}{2} + (1-\alpha)r.$$

Since $1 < \alpha = 1.618\cdots < 2$ we have $0 < \alpha - 1 < \frac{\alpha}{2} < 1$ and the result follows.

We now prove the theorem. Let the first row of the array consist of the Fibonacci numbers $1, 2 = f(1)$, $3 = f(2)$, $5 = f(3)$, $8 = f(5)$, and so on. The first positive integer not in this row is 4; let the second row be $4, 6 = f(4)$, $10 = f(6)$, $16 = f(10)$, and so on. The first positive integer not in the first or second row is 7; let the third row be $7, 11 = f(7)$, $18 = f(11)$, and so on. We see by Lemma 1 that there is no repetition. By Lemma 2 each row has the Fibonacci property. Finally, this process cannot terminate after a finite number of steps since the distances between successive elements in a row increase without bound. This completes the proof.

For the array just constructed, let a_n be the n^{th} number in the first column and b_n the n^{th} number in the second column. I conjecture that for $n \geq 2$ the difference $b_n - a_n$ is either a_i or b_i for some $i < n$.

We comment that the fact that $F_{n+1} = [\alpha F_n + \frac{1}{2}]$, where F_n is the n^{th} Fibonacci number, is Theorem III on p. 34 of the book *Fibonacci and Lucas Numbers*, Verner E. Hoggatt, Jr., Houghton Mifflin, Boston, 1969.

****