A PROBLEM OF FERMAT AND THE FIBONACCI SEQUENCE

V. E. HOGGATT, JR.
San Jose State University, San Jose, California 95192
and
G. E. BERGUM
South Dakota State University, Brookings, South Dakota 57007

1. INTRODUCTION

Fermat observed that the numbers 1, 3, 8, 120 have the following property: The product of any two increased by one is a perfect square. Davenport showed that for 1, 3, 8, x to have the same property x must be 120 and that it is impossible to find integers x and y such that the five numbers 1, 3, 8, x, y have this unique property.

In [1] and [2], B. W. Jones extends the problem to polynomials by showing

Theorem 1.1. Let $w^2 - 2(x+1)w + 1 = 0$ have $\alpha(x)$ and $\beta(x)$ as roots. Let $f_k(x) = (\alpha^k - \beta^k)/(\alpha - \beta)$. Let $c_k(x) = 2f_k(x)f_{k+1}(x)$. Then the polynomials x, $x+2$, $c_k(x)$, $c_{k+1}(x)$ have the property that the product of any two plus one is a perfect square.

Any enthusiast of the sequence of Fibonacci numbers would quickly observe that 1, 3, and 8 are terms of that sequence whose subscripts are consecutive even integers. That is, they are respectively F_2, F_4, and F_6. Using the Binet formula it is easy to show that the property enjoyed by 1, 3, and 8 is shared with any three terms of the Fibonacci sequence whose subscripts are consecutive even integers. In fact, we have

(1.1) $$F_{2n}F_{2n+2} + 1 = F_{2n+1}^2$$

(1.2) $$F_{2n}F_{2n+4} + 1 = F_{2n+2}^2$$

and

(1.3) $$F_{2n+2}F_{2n+4} + 1 = F_{2n+3}^2 .$$

One might now ask if there exists an integer x such that $F_{2n}x + 1$, $F_{2n+2}x + 1$ and $F_{2n+4}x + 1$ are perfect squares. In order to show that the answer is yes we proceed as follows. From (1.1) we see that

$$1 = F_{2n+1}^2 - F_{2n}F_{2n+2} = F_{2n+1}F_{2n+2} - F_{2n}F_{2n+3}$$

so that

(1.4) $$4F_{2n}F_{2n+1}F_{2n+2}F_{2n+3} + 1 = (2F_{2n+1}F_{2n+2} - 1)^2 .$$

Replacing n by $n+1$ in (1.1), we have

$$1 = F_{2n+3}^2 + F_{2n+1}F_{2n+4} - F_{2n+3}F_{2n+4} = F_{2n+1}F_{2n+4} - F_{2n+3}F_{2n+2}$$

so that

(1.5) $$4F_{2n+1}F_{2n+2}F_{2n+3}F_{2n+4} + 1 = (2F_{2n+2}F_{2n+3} + 1)^2 .$$

Using the Binet formula show that $F_{2n+2}^2 = F_{2n+1}F_{2n+3} - 1$. Multiply both sides of this equation by $4F_{2n+2}^2$ to obtain

(1.6) $$4F_{2n+1}F_{2n+2}^2F_{2n+3} + 1 = (2F_{2n+2}^2 + 1)^2.$$

Combining (1.1) through (1.6) we have

Theorem 1.2. For $n \geq 1$, the four numbers F_{2n}, F_{2n+2}, F_{2n+4}, and $x = 4F_{2n+1}F_{2n+2}F_{2n+3}$ have the property that the product of any two increased by one is a perfect square.

For n respectively 1, 2, and 3 we obtain the quadruples (1, 3, 8, 120), the result of Fermat, (3, 8, 21, 2080) and (8, 21, 55, 37128). The authors conjecture that the value x of Theorem 1.2 is unique.

Although three terms of the Fibonacci sequence whose subscripts are consecutive odd numbers do not have the property of those with even subscripts we do have the following.

Theorem 1.3. Let $n \geqslant 1$ and $x = 4F_{2n+2}F_{2n+3}F_{2n+4}$ then the numbers F_{2n+1}, F_{2n+3}, F_{2n+5} and x are such that

$$\begin{aligned} F_{2n+1}F_{2n+3} - 1 &= F_{2n+2}^2 \\ F_{2n+1}F_{2n+5} - 1 &= F_{2n+3}^2 \\ F_{2n+3}F_{2n+5} - 1 &= F_{2n+4}^2 \\ F_{2n+1}x + 1 &= (2F_{2n+2}F_{2n+3} + 1)^2 \\ F_{2n+3}x + 1 &= (2F_{2n+3}^2 - 1)^2 \\ F_{2n+5}x + 1 &= (2F_{2n+3}F_{2n+4} - 1)^2 . \end{aligned}$$

Here again the authors conjecture that the value of x in Theorem 1.3 is unique. Letting n respectively be 1, 2, and 3 in Theorem 1.3 we obtain the quadruples (2, 5, 13, 480), (5, 13, 34, 8136), and (13, 34, 89, 157080).

We now turn our attention to several problems which arose in our investigation of the results of Theorems 1.2 and 1.3.

First we wanted to know if there exists an x such that

$$\begin{cases} F_{2n}x - 1 = P^2 \\ F_{2n+2}x - 1 = M^2 \\ F_{2n+4}x - 1 = N^2 . \end{cases}$$

If such an x exists then by eliminating that value between pairs of equations, we have

$$\begin{cases} F_{2n}M^2 - F_{2n+2}P^2 = F_{2n+1} \\ F_{2n}N^2 - F_{2n+4}P^2 = L_{2n+2} \\ F_{2n+2}N^2 - F_{2n+4}M^2 = F_{2n+3} , \end{cases}$$

where L_i is the i^{th} Lucas number. One and only one of F_{2n}, F_{2n+1}, F_{2n+2} is even. Furthermore there exists an integer k such that $n = 3k$, $n = 3k + 1$, or $n = 3k + 2$. If $n = 3k$ then P is odd and the first equation becomes $-1 \equiv -F_{6k+2} \equiv F_{6k+1} \equiv 1 \pmod 4$ which is impossible. If $n = 3k + 1$ the first equation becomes $F_{6k+2}M^2 - F_{6k+4}P^2 = F_{6k+3}$. Since F_{6k+3} is even either M and P are both even or both odd. If both are even then $0 \equiv F_{6k+3} \equiv 2 \pmod 4$ which is impossible. If both are odd then $-2 \equiv F_{6k+2} - F_{6k+4} \equiv 2 \pmod 8$ which is impossible. When $n = 3k + 2$ M is odd and the first equation becomes $3 \equiv F_{6k+4} \equiv F_{6k+5} \equiv 1 \pmod 4$ which is impossible. Hence, the first equation is never solvable. Therefore no x can be found which satisfies the original system of equations. Following an argument similar to that given above it is easy to show that

$$F_{2n}N^2 - F_{2n+4}P^2 = L_{2n+2}$$

is impossible.

Next we tried to determine if more than one solution exists for

$$\text{(A)} \qquad \begin{cases} F_{2n}x + 1 = P^2 \\ F_{2n+2}x + 1 = M^2 \\ F_{2n+4}x + 1 = N^2 \end{cases} .$$

By eliminating the x we see that a necessary condition for a solution is

$$\text{(A}'\text{)} \qquad \begin{cases} F_{2n}M^2 - F_{2n+2}P^2 = -F_{2n+1} \\ F_{2n}N^2 - F_{2n+4}P^2 = -L_{2n+2} \\ F_{2n+2}N^2 - F_{2n+4}M^2 = -F_{2n+3} \end{cases} .$$

Recognizing that the first and last equations of (A′) are essentially the same, we conclude that a necessary condition for (A) to be solvable is that there exist a common solution of the Diophantine equations of the form

(1.7) $$F_{2n}M^2 - F_{2n+2}P^2 = -F_{2n+1}$$

and

(1.8) $$F_{2n}N^2 - F_{2n+4}P^2 = -L_{2n+2}.$$

Because of the relationships that exist between Diophantine equations of the form $Ax^2 - By^2 = \pm C$, continued fractions, and linear recurrences, we were led to consider the auxiliary polynomials

(1.9) $$w^2 - 2F_{2n+1}w + 1 = 0$$

and

(1.10) $$w^2 - 2F_{2n+2}w + 1 = 0.$$

Using these auxiliary polynomials we will develop a sequence of solutions to (1.7) and (1.8). In this and future developments we need the following lemma all of whose parts can be verified by using the Binet formula or formulas found in [1], [2].

Lemma 1.1. For all $k \geqslant 1$

(a) $$F_kF_{k+3}^2 - F_{k+2}^3 = (-1)^{k+1}F_{k+1}.$$

(b) $$F_{k+3}F_k^2 - F_{k+1}^3 = (-1)^{k+1}F_{k+2}.$$

(c) $$F_kF_{k+3}^2 - F_{k+4}F_{k+1}^2 = (-1)^{k+1}L_{k+2}.$$

(d) $$F_kL_{k+3}^2 - F_{k+4}L_{k+1}^2 = (-1)^{k+1}L_{k+2}.$$

2. SOLUTIONS OF $F_{2n}M^2 - F_{2n+2}P^2 = -F_{2n+1}$

We first turn our attention to (1.9) whose roots throughout this section are denoted by

$$\alpha = F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} \qquad \text{and} \qquad \beta = F_{2n+1} - \sqrt{F_{2n+1}^2 - 1}$$

Let $H_m = (\alpha^m - \beta^m)/(\alpha - \beta)$ then $\{H_m\}_{m=0}^{\infty}$ is given by

(2.1) $$H_0 = 0,\ H_1 = 1,\ H_m = 2F_{2n+1}H_{m-1} - H_{m-2}, \qquad m \geqslant 2$$

and it can be verified that

(2.2) $$H_{m-1}^2 - H_mH_{m-2} = 1.$$

With $M_m = AH_m + BH_{m-1}$, $P_m = A^*H_m + B^*H_{m-1}$ and (2.1), we see that

(2.3) $$\begin{cases} M_m = -M_1H_{m-2} + M_2H_{m-1} \\ P_m = -P_1H_{m-2} + P_2H_{m-1}. \end{cases}$$

Requiring that (M_m, P_m) be a solution of (1.7), provided (M_1, P_1) and (M_2, P_2) are, we have

$$(H_{m-2}^2 + H_{m-1}^2 - 1)F_{2n+1} = 2H_{m-1}H_{m-2}(F_{2n+2}P_1P_2 - F_{2n}M_1M_2)$$

which by using (2.1) and (2.2) becomes

(2.4) $$F_{2n+1}^2 = F_{2n+2}P_1P_2 - F_{2n}M_1M_2.$$

Obviously (±1, ±1) is a solution of (1.7) and Lemma 1.1, part (a), tells us that $(\pm F_{2n+3}, \pm F_{2n+2})$ is also a solution. Checking the sixteen possible combinations respectively for (M_1, P_1) and (M_2, P_2) in (2.4) we find only four solutions which are

$$\{(1,1), (F_{2n+3}, F_{2n+2})\}, \quad \{(1,-1), (F_{2n+3}, -F_{2n+2})\}, \quad \{(-1,1), (-F_{2n+3}, F_{2n+2})\},$$

and

$$\{(-1,-1), \quad (-F_{2n+3}, -F_{2n+2})\}.$$

Each of these four solutions, when used in conjunction with (2.3), gives us a sequence $\{(M_m, P_m)\}_{m=1}^{\infty}$ of solutions to (1.7) which, except for signs, are the same. Because of the exponents in (1.7) we consider only those pairs given by

(2.5) $$\begin{cases} M_m = 2F_{2n+1}M_{m-1} - M_{m-2} \\ P_m = 2F_{2n+1}P_{m-1} - P_{m-2} \end{cases}$$

where $M_1 = P_1 = 1$, $M_2 = F_{2n+3}$, and $P_2 = F_{2n+2}$.

Noting that the auxiliary polynomial for $\{M_m\}_{m=1}^{\infty}$ and $\{P_m\}_{m=1}^{\infty}$ is $w^2 - 2F_{2n+1}w + 1 = 0$, it is easy to show by standard techniques that

(2.6) $$\begin{cases} M_m = [(F_{2n}\beta + 1)\alpha^m - (F_{2n}\alpha + 1)\beta^m]/(\alpha - \beta) \\ P_m = [(-F_{2n-1}\beta + 1)\alpha^m - (-F_{2n-1}\alpha + 1)\beta^m]/(\alpha - \beta) \end{cases}$$

Let $D_M = M_m^2 - M_{m-1}M_{m+1}$ be the characteristic of $\{M_m\}_{m=1}^{\infty}$. Using (2.6), it can be shown that

(2.7) $$D_M = F_{2n+1}F_{2n+2} \quad \text{and} \quad D_P = -F_{2n}F_{2n+1}$$

or

(2.8) $$F_{2n}D_M = -F_{2n+2}D_P\ .$$

Using

(2.9) $$\begin{cases} M_{m-2} = 2F_{2n+1}M_{m-1} - M_m \\ P_{m-2} = 2F_{2n+1}P_{m-1} - P_m \end{cases}$$

together with part (b) of Lemma 1.1 it can be verified that $\{(\overline{M}_m, \overline{P}_m)\}_{m=1}^{\infty}$ is another sequence of solutions of (1.7) where

(2.10) $$\begin{cases} \overline{M}_m = 2F_{2n+1}\overline{M}_{m-1} - \overline{M}_{m-2} \\ \overline{P}_m = 2F_{2n+1}\overline{P}_{m-1} - \overline{P}_{m-2} \end{cases}$$

with

$$\overline{M}_1 = \overline{P}_1 = 1, \quad \overline{M}_2 = -F_{2n} \quad \text{and} \quad \overline{P}_2 = F_{2n-1}\ .$$

The sequences $\{\overline{M}_m\}_{m=1}^{\infty}$ and $\{\overline{P}_m\}_{m=1}^{\infty}$ are called conjugate sequences of $\{M_m\}_{m=1}^{\infty}$ and $\{P_m\}_{m=1}^{\infty}$. Since the auxiliary polynomial for $\{\overline{M}_m\}_{m=1}^{\infty}$ and $\{\overline{P}_m\}_{m=1}^{\infty}$ is

$$w^2 - 2F_{2n+1} + 1 = 0,$$

we see by standard techniques that

(2.11) $$\begin{cases} \overline{M}_m = [(-F_{2n+3}\beta + 1)\alpha^m - (-F_{2n+3}\alpha + 1)\beta^m]/(\alpha - \beta) \\ \overline{P}_m = [(-F_{2n+2}\beta + 1)\alpha^m - (-F_{2n+2}\alpha + 1)\beta^m]/(\alpha - \beta) \end{cases}$$

(2.12) $$D_{\overline{M}} = F_{2n+1}F_{2n+2} = D_M$$

and

(2.13) $$D_{\overline{P}} = -F_{2n}F_{2n+1} = D_P\,.$$

3. SOLUTIONS OF $F_{2n}N^2 - F_{2n+4}P^2 = -L_{2n+2}$

We now turn our attention to (1.10) whose roots throughout this section are denoted by

$$\gamma = F_{2n+2} + \sqrt{F_{2n+2}^2 - 1} \quad \text{and} \quad \delta = F_{2n+2} - \sqrt{F_{2n+2}^2 - 1}\ .$$

Let

$$H_m = (\gamma^m - \delta^m)/(\gamma - \delta)$$

then $\{H_m\}_{m=0}^{\infty}$ is given by

(3.1) $$H_0 = 0, \quad H_1 = 1, \quad H_m = 2F_{2n+2}H_{m-1} - H_{m-2}, \quad m \geqslant 2$$

and it can be shown that

(3.2) $$H_{m-1}^2 - H_mH_{m-2} = 1.$$

Let (N_1, P_1), (N_2, P_2) be solutions of (1.8). Let $\{(N_m, P_m)\}_{m=3}^{\infty}$ be given by

(3.3)
$$\begin{cases} N_m = -N_1H_{m-2} + N_2H_{m-1} \\ P_m = -P_1H_{m-2} + P_2H_{m-1} \end{cases}.$$

Let (N_m, P_m) be a solution of (1.8). By an argument similar to that given in Section 2, we find

(3.4)
$$L_{2n+2}F_{2n+2} = F_{2n+4}P_1P_2 - F_{2n}N_1N_2.$$

Lemma 1.1, part (c), yields $(\pm F_{2n+3}, \pm F_{2n+1})$ as a solution of (1.8). Obviously $(\pm 1, \pm 1)$ is a solution of (1.8). Letting these pairs be (N_1, P_1) and (N_2, P_2) we obtain sixteen possible values for (3.4). Using

(3.5)
$$L_{2n+2}F_{2n+2} = F_{2n+4}F_{2n+1} + F_{2n}F_{2n+3}$$

it is easy to check that only four solutions exist which, except for signs, are the same. The solution we use gives rise to

(3.6)
$$\begin{cases} N_m = 2F_{2n+2}N_{m-1} - N_{m-2} \\ P_m = 2F_{2n+2}P_{m-1} - P_{m-2} \end{cases},$$

where

$$N_1 = P_1 = 1, \quad N_2 = -F_{2n+3} \quad \text{and} \quad P_2 = F_{2n+1}.$$

Furthermore,

(3.7)
$$\begin{cases} N_m = [(-L_{2n+3}\delta + 1)\gamma^m - (-L_{2n+3}\gamma + 1)\delta^m]/(\gamma - \delta) \\ P_m = [(-L_{2n+1}\delta + 1)\gamma^m - (-L_{2n+1}\gamma + 1)\delta^m]/(\gamma - \delta) \end{cases}$$

(3.8)
$$D_N = F_{2n+4}L_{2n+2}, \quad D_P = -F_{2n}L_{2n+2}$$

and

(3.9)
$$F_{2n}D_N = -F_{2n+4}D_P.$$

The conjugate sequences $\{\bar N_m\}_{m=1}^{\infty}$ and $\{\bar P_m\}_{m=1}^{\infty}$ are given by

(3.10)
$$\begin{cases} \bar N_m = 2F_{2n+2}\bar N_{m-1} - \bar N_{m-2} \\ \bar P_m = 2F_{2n+2}\bar P_{m-1} - \bar P_{m-2} \end{cases}$$

with

$$\bar N_1 = \bar P_1 = 1, \quad \bar N_2 = L_{2n+3}, \quad \text{and} \quad \bar P_2 = L_{2n+1}.$$

Using Lemma 1.1, part (d), it can be shown that $\{(\bar N_m, \bar P_m)\}_{m=1}^{\infty}$ is a sequence of solutions to (1.8). Furthermore,

(3.11)
$$\begin{cases} \bar N_m = [(F_{2n+3}\delta + 1)\gamma^m - (F_{2n+3}\gamma + 1)\delta^m]/(\gamma - \delta) \\ \bar P_m = [(-F_{2n+1}\delta + 1)\gamma^m - (-F_{2n+1}\gamma + 1)\delta^m]/(\gamma - \delta) \end{cases}$$

(3.12)
$$D_{\bar N} = F_{2n+4}L_{2n+2} = D_N$$

and

(3.13)
$$D_{\bar P} = -F_{2n}L_{2n+2} = D_P.$$

Although the results of Sections 2 and 3 do not directly give a solution to (A), we can generate an infinite sequence of solutions for each of the equations of (A') by using (2.5), (2.10), (3.6) and (3.10).

4. CONCLUDING REMARKS

By eliminating the x value between pairs of equations we see that a necessary condition for

(B)
$$\begin{cases} F_{2n+1}x + 1 = R^2 \\ F_{2n+3}x + 1 = S^2 \\ F_{2n+5}x + 1 = T^2 \end{cases}$$

or

(C)
$$\begin{cases} F_{2n+1}x - 1 = R^2 \\ F_{2n+3}x - 1 = S^2 \\ F_{2n+5}x - 1 = T^2 \end{cases}$$

to be solvable is

(B′)
$$\begin{cases} F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2} \\ F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3} \\ F_{2n+3}T^2 - F_{2n+5}S^2 = -F_{2n+4} \end{cases}$$

or

(C)
$$\begin{cases} F_{2n+1}x - 1 = R^2 \\ F_{2n+3}x - 1 = S^2 \\ F_{2n+5}x - 1 = T^2 \end{cases}$$

to be solvable is

(B′)
$$\begin{cases} F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2} \\ F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3} \\ F_{2n+3}T^2 - F_{2n+5}S^2 = -F_{2n+4} \end{cases}$$

or

(C′)
$$\begin{cases} F_{2n+1}S^2 - F_{2n+3}R^2 = F_{2n+2} \\ F_{2n+1}T^2 - F_{2n+5}R^2 = L_{2n+3} \\ F_{2n+3}T^2 - F_{2n+5}S^2 = F_{2n+4} \end{cases}.$$

Recognizing the similarity of several of the equations we are led to consider only solutions of Diophantine equations of the form

(4.1)
$$F_{2n+1}S^2 - F_{2n+3}R^2 = \mp F_{2n+2}$$

and

(4.2)
$$F_{2n+1}T^2 - F_{2n+5}R^2 = \mp L_{2n+3}\ .$$

CASE I: $F_{2n+1}S^2 - F_{2n+3}R^2 = \mp F_{2n+2}$.

In this case we consider the auxiliary polynomial

$$w^2 - 2F_{2n+2}w - 1 = 0$$

whose roots are denoted by

$$\epsilon = F_{2n+2} + \sqrt{F_{2n+2}^2 + 1} \quad \text{and} \quad \sigma = F_{2n+2} - \sqrt{F_{2n+2}^2 + 1}\ .$$

Following the techniques of Section 2 it can be shown that

(4.3)
$$\begin{cases} S_m = 2F_{2n+2}S_{m-1} + S_{m-2} \\ R_m = 2F_{2n+2}R_{m-1} + R_{m-2} \end{cases},$$

with

$$S_1 = R_1 = 1, \quad S_2 = F_{2n+4}, \quad \text{and} \quad R_2 = F_{2n+3},$$

is a solution of

$$F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2}$$

when m is odd and

$$F_{2n+1}S^2 - F_{2n+3}R^2 = F_{2n+2}$$

when m is even. Furthermore

$$\begin{cases} S_m = [(-F_{2n+1}\sigma + 1)\epsilon^m - (-F_{2n+1}\epsilon + 1)\sigma^m]/(\epsilon - \sigma) \\ R_m = [(F_{2n}\sigma + 1)\epsilon^m - (F_{2n}\epsilon + 1)\sigma^m]/(\epsilon - \sigma) \end{cases} \tag{4.4}$$

$$D_S = (-1)^m F_{2n+3}F_{2n+2} \qquad D_R = (-1)^{m+1}F_{2n+1}F_{2n+2} \tag{4.5}$$

and

$$F_{2n+3}D_R = -F_{2n+1}D_S. \tag{4.6}$$

The conjugate sequences $\{\overline{S}_m\}_{m=1}^{\infty}$ and $\{\overline{R}_m\}_{m=1}^{\infty}$ are given by

$$\begin{cases} \overline{S}_m = -2F_{2n+2}\overline{S}_{m-1} + \overline{S}_{m-2} \\ \overline{R}_m = -2F_{2n+2}\overline{R}_{m-1} + \overline{R}_{m-2} \end{cases} \tag{4.7}$$

with

$$\overline{S}_1 = \overline{R}_1 = 1, \qquad \overline{S}_2 = F_{2n+1} \quad \text{and} \quad \overline{R}_2 = -F_{2n}.$$

When m is odd, $(\overline{S}_m, \overline{R}_m)$ is a solution of

$$F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2}$$

while it is a solution of

$$F_{2n+1}S^2 - F_{2n+3}R^2 = F_{2n+2}$$

when m is even.

Furthermore

$$\begin{cases} \overline{S}_m = [(F_{2n+4}\epsilon + 1)(-\sigma)^m - (F_{2n+4}\sigma + 1)(-\epsilon)^m]/(\epsilon - \sigma) \\ \overline{R}_m = [(F_{2n+3}\epsilon + 1)(-\sigma)^m - (F_{2n+3}\sigma + 1)(-\epsilon)^m]/(\epsilon - \sigma) \end{cases} \tag{4.8}$$

$$D_{\overline{S}} = D_S = (-1)^m F_{2n+3}F_{2n+2} \tag{4.9}$$

and

$$D_{\overline{R}} = D_R = (-1)^{m+1}F_{2n+1}F_{2n+2}. \tag{4.10}$$

CASE II: $F_{2n+1}T^2 - F_{2n+5}R^2 = \mp L_{2n+3}$.

In this case we consider the auxiliary polynomial $w^2 - 2F_{2n+3}w - 1 = 0$ whose roots are

$$\psi = F_{2n+3} + \sqrt{F_{2n+3}^2 + 1} \quad \text{and} \quad \xi = F_{2n+3} - \sqrt{F_{2n+3}^2 + 1}.$$

Following the techniques of Section 2, it can be shown that

$$\begin{cases} T_m = 2F_{2n+3}T_{m-1} + T_{m-2} \\ R_m = 2F_{2n+3}R_{m-1} + R_{m-2} \end{cases}, \tag{4.11}$$

with

$$T_1 = R_1 = 1, \qquad T_2 = -F_{2n+4}, \quad \text{and} \quad R_2 = F_{2n+2},$$

is a solution of

$$F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3}$$

when m is odd and

$$F_{2n+1}T^2 - F_{2n+5}R^2 = L_{2n+3}$$

when m is even. Furthermore

$$\begin{cases} T_m = [(L_{2n+4}\xi + 1)\psi^m = (L_{2n+4}\psi + 1)\xi^m]/(\psi - \xi) \\ R_m = [(L_{2n+2}\xi + 1)\psi^m = (L_{2n+2}\psi + 1)\xi^m]/(\psi - \xi) \end{cases} \tag{4.12}$$

(4.13) $$D_T = (-1)^m F_{2n+5} L_{2n+3}, \qquad D_R = (-1)^{m+1} F_{2n+1} L_{2n+3}$$

and

(4.14) $$F_{2n+1} D_T = -F_{2n+5} D_R\ .$$

The conjugate sequences $\{\overline{T}_m\}_{m=1}^{\infty}$ and $\{\overline{R}_m\}_{m=1}^{\infty}$ are given by

(4.15) $$\begin{cases} \overline{T}_m = -2F_{2n+3}\overline{T}_{m-1} + \overline{T}_{m-2} \\ \overline{R}_m = -2F_{2n+3}\overline{R}_{m-1} + \overline{R}_{m-2} \end{cases}$$

with

$$\overline{T}_1 = \overline{R}_1 = 1, \qquad \overline{T}_2 = -L_{2n+4} \quad \text{and} \quad \overline{R}_2 = -L_{2n+2}\,.$$

When m is odd $(\overline{T}_m, \overline{R}_m)$ is a solution of

$$F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3}$$

while it is a solution of

$$F_{2n+1}T^2 - F_{2n+5}R^2 = L_{2n+3}$$

when m is even. Furthermore

(4.8) $$\begin{cases} \overline{T}_m = [(-F_{2n+4}\psi + 1)(-\xi)^m - (-F_{2n+4}\xi + 1)(-\psi)^m]/(\psi - \xi) \\ \overline{R}_m = [(F_{2n+2}\psi + 1)(-\xi)^m - (F_{2n+2}\xi + 1)(-\psi)^m]/(\psi - \xi) \end{cases}$$

(4.9) $$D_{\overline{T}} = D_T = (-1)^m F_{2n+5} L_{2n+3}$$

and

(4.10) $$D_{\overline{R}} = D_R = (-1)^{m+1} F_{2n+1} L_{2n+3}\,.$$

In closing, we observe that if you choose $m = 3$ in (2.5) and (3.6) you obtain

$M_3 = 2F_{2n+1}F_{2n+3} - 1 = 2F_{2n+2}^2 + 1,\quad P_3 = 2F_{2n+1}F_{2n+2} - 1,$ and $N_3 = -2F_{2n+2}F_{2n+3} - 1$

which are equivalent to the values in (1.6), (1.4), and (1.5). Letting $m = 3$ in (4.3) and (4.11) you obtain

$S_3 = 2F_{2n+2}F_{2n+4} + 1 = 2F_{2n+3}^2 - 1,\quad R_3 = 2F_{2n+2}F_{2n+3} + 1,$ and $T_3 = -2F_{2n+3}F_{2n+4} + 1$

which are equivalent to the values in Theorem 1.3.

REFERENCES

1. B. W. Jones, "A Variation on a Problem of Davenport and Diophantus," *Quarterly J. of Math.,* Vol. 27 (1976), pp. 349–353.
2. B. W. Jones, "A Second Variation on a Problem of Diophantus," *The Fibonacci Quarterly,* to appear.

****