GENERALIZED QUATERNIONS WITH QUATERNION COMPONENTS

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The relations connecting generalized Fibonacci Quaternions obtained by Iyer [3], following earlier work by Horadam [2], together with the recent article by Swamy [4], prompted this note on further generalized quaternions, as well as an investigation of generalized quaternions whose components are quaternions.

Following the ideas of [3] we define

1. (a) $$P_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}$$
(b) $$Q_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}$$
(c) $$R_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3},$$

where

(d) $$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k$$
$$jk = -kj = i, \quad ki = -ik = j$$

and where

2. (a) $$W_n = pW_{n-1} - qW_{n-2} \qquad W_0 = a, \quad W_1 = b$$
(b) $$U_n = pU_{n-1} - qU_{n-2} \qquad U_0 = 1, \quad U_1 = p$$
(c) $$V_n = pV_{n-1} - qV_{n-2} \qquad V_0 = 2, \quad V_1 = p.$$

Thus from 1(a) and 2(a) we have that

3. $$P_n = pP_{n-1} - qP_{n-2}.$$

Analogous results to equations 2.14 and 2.15 of Horadam [1] are, respectively:

4. $$P_n = aQ_n + (b - pa)Q_{n-1}$$
5. $$R_n = 2Q_n - pQ_{n-1}.$$

The conjugate quaternion of P_n is given by

6. $$\bar{P}_n = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$$

We now define the quaternions T_n and S_n as the quaternions whose components are the quaternions P_n and Q_n, respectively, viz.

7. (a) $$T_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$
(b) $$S_n = Q_n + iQ_{n+1} + jQ_{n+2} + kQ_{n+3}$$

which on expanding give

8. (a) $$T_n = W_n - W_{n+2} - W_{n+4} - W_{n+6} + 2iW_{n+1} + 2jW_{n+2} + 2kW_{n+3}$$

and similarly for S_n.

The conjugate for T_n is

9. (a) $$\bar{T}_n = P_n - iP_{n+1} - jP_{n+2} - kP_{n+3}$$

which becomes on expansion

(b) $$\bar{T}_n = W_n + W_{n+2} + W_{n+4} + W_{n+6}$$

so that the conjugate quaternion can be expressed solely in terms of W_n's and is independent of the vectors i, j, k.

Now consider

$$Q_{-n} = U_{-n} + iU_{-n+1} + jU_{-n+2} + kU_{-n+3} .$$

Using equation 2.17 of [1] and noting that the result should be

$$U_{-n} = -q^{-n+1}U_{n-2}$$

we obtain

$$Q_{-n} = -q^{-n+1}[U_{n-2} + iqU_{n-3} + jq^2U_{n-4} + kq^3U_{n-5}]$$

10. $$Q_{-n} = -q^{-n+1}Q^*_{n-2},$$

where we define

11. $$Q^*_n = U_n + iqU_{n-1} + jq^2U_{n-2} + kq^3U_{n-3} .$$

Similarly we have that

12. $$Q^*_{-n} = -q^{-n+1}Q_{n-2} .$$

Using the above we shall now establish some relations between these quaternions. The first of these is

13. $$P_nP_{n+t} + eq^{n-r}Q_{r-1}Q_{r+t-1} = W_{n-r}T_{n+r+t} .$$

The proof for this is lengthy and is left to the reader. A direct proof uses 1(a), 1(b), 7(a) and equation 4.18 of Horadam [1].

Now letting $t = 0$ in equation (13) above we have

14. $$P^2_n + eq^{n-r}Q^2_{r-1} = W_{n-r}T_{n+r} .$$

If we let $r = 1$ in equation (14) we obtain

15. $$eq^{n-1}\sum_{j=0}^{3} U_j^2 = P_n^2 + 2eq^{n-1}Q_0 - W_{n-1}T_{n+1} .$$

Another identity is

16. $$aP_{m+n} + (b-pa)P_{m+n-1} = W_mP_n - qW_{m-1}P_{n-1} .$$

The proof uses 1(a) and equation 4.1 of Horadam [1].

Further results are

17. $$P_mP_n - qP_{m-1}P_{n-1} = aT_{m+n} + (b-pa)T_{m+n-1} = W_mT_n - qW_{m-1}T_{n-1} .$$

For $m = n$ in (17)

18. $$P_n^2 - qP_{n-1}^2 = aT_{2n} + (b-pa)T_{2n-1} = W_nT_n - qW_{n-1}T_{n-1}$$

19. $$P^2_{n+1} - q^2P_{n-1} = bT_{2n+1} + (b-pa)qT_{2n-1}$$

20. $$bP_{2n+1} + (b-pa)qP_{2n-1} = W_{n+1}P_{n+1} - q^2W_{n-1}P_{n-1} .$$

Now from 7(b)

21. (a) $$2S_{m+n-1} = R_nQ_{m-1} + Q_{n-1}R_m$$

(b) $$2Q_{m+n-1} = U_{m-1}R_n + Q_{n-1}V_m = Q_{m-1}V_n + U_{n-1}R_m$$

22. (a) $$P_{n+r} = U_nP_r - qU_{n-1}P_{r-1} = W_nQ_r - qW_{n-1}Q_{r-1}$$

(b) $$T_{n+r} = P_nQ_r - qP_{n-1}Q_{r-1} = U_nT_r - qU_{n-1}T_{r-1} = W_nS_r - qW_{n-1}S_{r-1}$$

23. $$2R_{m+n} = V_mR_n + d^2U_{m-1}Q_{n-1},$$

where $d^2 = p^2 - 4q$.

24. (a) $$P_{n+r} + q^rP_{n-r} = P_nV_r$$

(b) $$T_{n+r} + q^rT_{n-r} = T_nV_r$$

Now recalling the notation we established in equation (11) we let

25. $$P^*_n = W_n + iqW_{n-1} + jq^2W_{n-2} + kq^3W_{n-3} .$$

We are thus able to establish the interesting relations

26. $$P_{n-r}P_{n+r+t} - P_nP_{n+t} = eq^{n-r}U_{r-1}S^*_{r+t-1}$$

27. $$P_{n-r}P_{n+r+t} - P_{n+t}P_n = eq^{n-r}U_{r+t-1}S^*_{r-1}\,.$$

Thus we note the change in the R.H.S. expressions for equations (26) and (27) when the only difference in the L.H.S. is that the elements in the subtracted product term have been commuted. This is to be expected as quaternion multiplication is non-commutative.

Similarly we obtain

28. (a) $$P_{n-r}T_{n+r+t} - P_nT_{n+t} = eq^{n-r}U_{r-1}(S_{r+t-1} + iqS_{r+t-2} + jq^2S_{r+t-3} + kq^3S_{r+t-4})$$

(b) $$P_{n-r}T_{n+r+t} - P_{n+t}T_n = eq^{n-r}U_{r+t-1}(S_{r-1} + iqS_{r-2} + jq^2S_{r-3} + kq^3S_{r-4})$$

29. $$P_{m-r}P_{n+r} - P_{n-r}P_{m+r} = eq^{m-r}U_{n-m-1}S^*_{2r-1}$$

and where $e = pab - qa^2 - b^2$ from equation (2).

At this point it is interesting to note the correlation of the above equations (13), (14), (16), (17), (18), (19), (20), (21), (22), (23), (24) and $\{(26), (27), (28)\}$ with equations 4.18, 4.5, 4.1, 4.1, 4.2, 4.17, 4.17, 4.8, 3.14, 4.9, 3.16, 4.18 of Horadam [1], respectively. The equations listed from Horadam were in fact used to obtain the corresponding results for the generalized quaternions.

From 9(b) we have for the conjugate quaternion $\bar{T}_{2n}$

$$\bar{T}_{2n} = W_{2n} + W_{2n+2} + W_{2n+4} + W_{2n+6}$$

and thus

$$a\bar{T}_{2n} = aW_{2n} + aW_{2n+2} + aW_{2n+4} + aW_{2n+6}\,.$$

Using equation 4.5 of Horadam [1] we have

$$a\bar{T}_{2n} = W_n^2 + W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 + e(U_{n-1}^2 + U_n^2 + U_{n+1}^2 + U_{n+2}^2)$$

but

$$P_n^2 = W_n^2 - W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2 + 2iW_nW_{n+1} + 2jW_nW_{n+2} + 2kW_nW_{n+3}$$

and similarly for Q_n^2.

Therefore

30. $$a\bar{T}_{2n} + P_n^2 + eQ_{n-1}^2 = 2(W_nP_n + eU_{n-1}Q_{n-1})\,.$$

Many more results can be obtained for the above-defined quaternions. By use of a functional notation the ideas expressed in this article can be easily extended.

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