ROW AND RISING DIAGONAL SUMS FOR A TYPE OF PASCAL TRIANGLE

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As has been noted by Hoggatt [1], Pascal's Triangle can be thought of as having been generated by column generators. This provides insight into the row sums and rising diagonal sums of this triangle. Let $\{a_{i,0}\}_{i=0}^{\infty}$ denote a real number sequence and consider the following array:

$$\begin{array}{cccccccc} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,m} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & & a_{n,m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{array}$$

which has the Pascal-like property [3]

$$a_{ij} = \begin{cases} a_{i-1,j-1} + a_{i-1,j} & \text{if } i \geqslant j \geqslant 1, \\ 0 & \text{if } j > i \geqslant 0. \end{cases}$$

Under these conditions, it follows readily that

$$a_{ij} = \sum_{k=0}^{i-1} a_{k,j-1}$$

for all i and j such that $i \geqslant j \geqslant 1$. For the following assume that $f(x)$ is the generating function for the sequence $\{a_{i,0}\}_{i=1}^{\infty}$.

Theorem 1. The generating function for the k^{th} column $(k = 0,1,2,\cdots)$ in the above array is

$$g_k(x) = f(x)[x/(1-x)]^k.$$

Proof. Let

$$f(x) = a_{0,0} + a_{1,0}x + a_{2,0}x^2 + \cdots$$

denote the generating function for the zeroth column $\{a_{i,0}\}_{i=1}^{\infty}$. Suppose that

$$f(x)[x/(1-x)]^m = \sum_{i=0}^{\infty} (a_{i,m})x^i$$

for some positive integer m. Then by the comment preceding Theorem 1

$$f(x)[x/(1-x)]^{m+1} = f(x)[x/(1-x)]^m[x/(1-x)] = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{i-1} a_{k,m}\right)x^i = \sum_{i=0}^{\infty}(a_{i,m+1})x^i$$

and the proof is complete by induction on m.

Theorem 2. The generating function for the row sums of the above array is

$$[f(x)(1-x)]/(1-2x).$$

Proof. Since $g_k(x) = f(x)[x/(1-x)]^k$, the generating function for the row sums is

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = f(x)\sum_{k=0}^{\infty}\left(\frac{x}{1-x}\right)^k = f(x)\left[\frac{1}{1-\dfrac{x}{1-x}}\right] = f(x)\,\frac{1-x}{1-2x}\,.$$

Theorem 3. The generating function for the rising diagonal sums of the above array is

$$[f(x)(1-x)]/(1-x-x^2).$$

Proof. Consider the new array:

$$\begin{array}{cccccc}
a_{0,0} & 0 & 0 & 0 & 0 & \cdots \\
a_{1,0} & a_{0,1} & 0 & 0 & 0 & \cdots \\
a_{2,0} & a_{1,1} & a_{0,2} & 0 & 0 & \cdots \\
a_{3,0} & a_{2,1} & a_{1,2} & a_{0,3} & 0 & \cdots \\
a_{4,0} & a_{3,1} & a_{2,2} & a_{1,3} & a_{0,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
f(x) & xg_1(x) & x^2g_2(x) & x^3g_3(x) & x^4g_4(x) & \cdots
\end{array}$$

Note that the column generator for the k^{th} column $(k = 0, 1, 2, \cdots)$ is $x^kg_k(x)$. Furthermore the row sums of this array are the rising diagonal sums of the original array. Thus the generating function for the rising diagonal sums in the original array is

$$D(x) = \sum_{k=0}^{\infty} x^kg_k(x) = f(x)\sum_{k=0}^{\infty}\left(\frac{x^2}{1-x}\right)^k = f(x)\,\frac{1-x}{1-x-x^2}$$

Now if $f(x) = 1/(1-x)$, one has the usual Pascal Triangle and some of results in [1]. Moreover, if

$$f(x) = x[(1-x)/(1-3x+x^2)] + 1 = (1-2x)/(1-3x+x^2),$$

then the theorems above can be used to answer Problem H-183 [4] in this *Quarterly.* Indeed, since

$$H(x) = \frac{a+(b-ap)x}{1-px+qx^2}$$

is the generating function for the generalized Fibonacci sequence $w_n = w_n(a,b;p,q)$ [2], and

$$O(x) = \frac{w_1 + [w_3 - (p^2-2q)w_1]x}{1-(p^2-2q)x+q^2x^2}$$

and

$$E(x) = \frac{w_0 + [w_2 - (p^2-2q)w_0]x}{1-(p^2-2q)x+q^2x^2}$$

are the generating functions for the sequences $\{w_{2k+1}\}_{k=1}^{\infty}$ and $\{w_{2k}\}_{k=1}^{\infty}$ respectively, then questions similar to H-183 can be answered readily by considering the generating functions $xH(x)+1$, $xO(x)+1$, and $xE(x)+1$. In particular, if one considers the sequence $\{a_{i,0}\}_{i=0}^{\infty}$ where $a_{0,0} = 1$ and $a_{i,0} = L_{2i-1}$ (for $i = 1, 2, 3, \cdots$), then the array is

$$\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
4 & 2 & 1 & & \\
11 & 6 & 3 & 1 & \\
29 & 17 & 9 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

the generating function for the zeroth column is

$$(1-2x+2x^2)/(1-3x+x^2),$$

the generating function for the row sums is

$$(1-3x+4x^2-2x^3)/(1-5x+7x^2-2x^3),$$

and the generating function for the rising diagonal sums is

$$(1-3x+4x^2)/(1-4x+3x^2+2x^3-x^4).$$

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3. J. G. Kemeny, H. Mirkil, J. L. Snell, and G. L. Thompson, *Finite Mathematical Structures,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1959, p. 93.
4. Problem H-183, *The Fibonacci Quarterly,* Vol. 9, No. 4 (1971), p. 389, by V. E. Hoggatt, Jr.

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In the paper Tribonacci Sequence by April Scott, Tom Delaney and V. E. Hoggatt,Jr. in the October FQJ the following references should be appended.

1. Mark Feinberg, Fibonacci-Tribonacci, FQJ Oct.,1963 pp71-74
2. Trudy Tong, Some Properties of the Tribonacci Seqquence and The Special Lucas Sequence, Unpublished Masters Thesis, San Jose State University, August,1970
3. C. C. Yalavigi, Properties of the Tribonacci Numbers FQJ Oct. 1972 pp231-246
4. Krishnaswami Alladi, On Tibonacci Numbers and Related Functions, FQJ Feb.,1977 pp42-46
5. A. G. Shannon, Tribonacci Numbers and Pascals Pyramid, FQJ Oct., 1977 pp 268+ 275

The term TRIBONACCI number was coined by Mark Feinberg in [1]above.