BINARY SEQUENCES WITHOUT ISOLATED ONES

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Liu [2] asks for the number of sequences of zeros and ones of length five, such that every digit 1 has at least one neighboring 1. The solution [1] uses the principle of inclusion-exclusion, although it is easier in this particular case to enumerate the twelve sequences:

00000, 11000, 01100, 00110, 00011, 11100, 01110, 00111, 11110, 01111, 11011, 11111.

In order to obtain a general result it seemed to us easier to find a recurrence relation.

Call a sequence *good* if each one in it has a neighboring one, and let a_n be the number of good sequences of length n. For example,

$$a_1 = 1,\quad a_2 = 2,\quad a_3 = 4,\quad a_4 = 7 \quad\text{and}\quad a_5 = 12.$$

Good sequences of length n are obtained from other good sequences of length $n-1$ by appending 0 or 1 to them, except that

(a) some not good sequences are also produced, namely those which end in 01, but are otherwise good, and
(b) there are good sequences which are not produced in this way; those obtained by appending 011 to a good sequence of length $n-3$.

So

(1) $$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}\ .$$

Alternatively, all good sequences are obtained from shorter good sequences by appending 0, 11 or 0111, so that

(2) $$a_n = a_{n-1} + a_{n-2} + a_{n-4}\,.$$

The characteristic equation for (2) is the same as that for (1), namely

(3) $$x^3 - 2x^2 + x - 1 = 0,$$

except for the additional root -1. The equation (3) has one real root, $\gamma \approx 1.754877666247$ and two complex roots, $\alpha \pm i\beta$, the square of whose modulus, $1/\gamma$, is less than 1.

$$a_n = c\gamma^n + (a+ib)(\alpha+i\beta)^n + (a-ib)(\alpha-i\beta)^n,$$

where

$$\alpha = 1 - \tfrac{1}{2}\gamma \approx 0.122561166876,\quad \beta = \sqrt{3\gamma^2 - 4\gamma}/2 \approx 0.744861766619,$$

$$a = (\gamma^2 - 2\gamma + 2)/2(2\gamma^2 - 2\gamma + 3) \approx 0.138937790848,\quad b = (2\gamma+1)(\gamma-1)/2\beta \approx 0.202250124098,$$

$$c = (\gamma^2+1)/(2\gamma^2 - 2\gamma + 3) \approx 0.722124418303\,,$$

and a_n is the nearest integer to $c\gamma^n$.

The sequence $\{a_n\}$ does not appear in Neil Sloane's book [3]; nor do the corresponding sequences $\{a_n^{(k)}\}$ of numbers of binary sequences of length n in which the ones occur only in blocks of length at least k. The problem so far considered is $k = 2$. The more general analogs of (1), (2), (3) are

(1′) $$a_n = 2a_{n-1} - a_{n-2} + a_{n-k-1}\,,$$

(2′) $$a_n = a_{n-1} + a_{n-k} + a_{n-k-2} + a_{n-k-3} + \cdots + a_{n-2k}\,,$$

(3′) $$x^{k+1} - 2x^k + x^{k-1} - 1 = 0.$$

Then

$$a_{-1}^{(k)} = a_0^{(k)} = a_1^{(k)} = \cdots = a_{k-1}^{(k)} = 1; \quad a_{k+r}^{(k)} = 1 + \tfrac{1}{2}(r+1)(r+2)$$

for $0 \leqslant r \leqslant k$; and for larger values of n, $a_n^{(k)}$ is the nearest integer to $c_k\gamma_k^n$, where γ_k is the real root of (3′) which lies between 1 and 2, and c_k is an appropriate constant. Approximate values of γ_k and c_k for $k = 1(1)9$ are shown in Table 1.

Table 1

k	1	2	3	4	5	6	7	8	9
γ_k	2	1.7549	1.6180	1.5289	1.4656	1.4178	1.3803	1.3499	1.3247
c_k	1	0.7221	0.5854	0.5033	0.4481	0.4082	0.3778	0.3539	0.3344

The sequence $\{a_n^{(3)}\}$ is similar to the Lucas sequence associated with the Fibonacci numbers, since $\gamma_3 = (1+\sqrt{5})/2$, the golden number.

The characteristic polynomial for (2′) is the product of that for (1′) with the cyclotomic polynomial $x^{k-1} + x^{k-2} + \cdots + x + 1$. When k is odd, (3′) is of even degree and is reducible and has a second real root between 0 and −1. Table 2 gives the values of $a_n^{(k)}$ for $n = 0(1)26$, $k = 2(1)9$. Of course, $a_n^{(1)} = 2^n$, the number of unrestricted binary sequences of length n.

Table 2

n	$a_n^{(2)}$	$a_n^{(3)}$	$a_n^{(4)}$	$a_n^{(5)}$	$a_n^{(6)}$	$a_n^{(7)}$	$a_n^{(8)}$	$a_n^{(9)}$
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1
3	4	2	1	1	1	1	1	1
4	7	4	2	1	1	1	1	1
5	12	7	4	2	1	1	1	1
6	21	11	7	4	2	1	1	1
7	37	17	11	7	4	2	1	1
8	65	27	16	11	7	4	2	1
9	114	44	23	16	11	7	4	2
10	200	72	34	22	16	11	7	4
11	351	117	52	30	22	16	11	7
12	616	189	81	42	29	22	16	11
13	1081	305	126	61	38	29	22	16
14	1897	493	194	91	51	37	29	22
15	3329	798	296	137	71	47	37	29
16	5842	1292	450	205	102	61	46	37
17	10252	2091	685	303	149	82	57	46
18	17991	3383	1046	443	218	114	72	56
19	31572	5473	1601	644	316	162	94	68
20	55405	8855	2452	936	452	232	127	84
21	97229	14328	3753	1365	639	331	176	107
22	170625	23184	5739	1999	897	467	247	141
23	299426	37513	8771	2936	1257	650	347	191
24	525456	60697	13404	4316	1766	894	484	263
25	922111	98209	20489	6340	2493	1220	667	364
26	1618192	158905	31327	9300	3536	1660	907	502

Since these are recurring sequences, they have many divisibility properties. Examples are $5 | a_n^{(2)}$ just if $n \equiv -4$ or -2, mod 12; $8 | a_n^{(2)}$ just if $n \equiv -4$ or -2, mod 14 and $2 | a_n^{(k)}$ according to the residue class to which n belongs, mod $2(2^{(k+1)/2} - 1)$, k odd, or mod $2^{k+1} - 1$, k even.

REFERENCES

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2. C.L. Liu, *Introduction to Combinatorial Mathematics,* McGraw-Hill, 1968, Problem 4–4, p. 119.
3. N. J. A. Sloane, *A Handbook of Integer Sequences,* Academic Press, 1973, p. 59.

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ON THE EQUALITY OF PERIODS OF DIFFERENT MODULI IN THE FIBONACCI SEQUENCE

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Let m be an arbitrary positive integer. According to the notation of Vinson [1, p. 37] let $s(m)$ denote the period of F_n modulo m and let $f(m)$ denote the rank of apparition of m in the Fibonacci sequence.

Let p be an arbitrary prime. Wall [2, p. 528] makes the following remark: "The most perplexing problem we have met in this study concerns the hypothesis $s(p^2) \neq s(p)$. We have run a test on a digital computer which shows that $s(p^2) \neq s(p)$ for all p up to 10,000; however, we cannot yet prove that $s(p^2) = s(p)$ is impossible. The question is closely related to another one, "can a number x have the same order mod p and mod p^2?," for which rare cases give an affirmative answer (e.g., $x = 3,\ p = 11;\ x = 2,\ p = 1093$); hence, one might conjecture that equality may hold for some exceptional p."

Based on Ward's Last Theorem [3, p. 205] we shall give necessary and sufficient conditions for $s(p^2) = s(p)$.

From Robinson [4, p. 30] we have for $m,n > 0$

(1) $$F_{n+r} \equiv F_r \pmod m \text{ for all integers } r \text{ if and only if } s(m)\,|\,n\,.$$

If $m,n > 0$ and $m\,|\,n$, then $F_{s(n)+r} \equiv F_r \pmod m$ for all r. Therefore by (1), $s(m)\,|\,s(n)$. So we have for $m,n > 0$

(2) $$m\,|\,n \text{ implies } s(m)\,|\,s(n)\,.$$

It is easily verified that for all integers n

(3) $$F_{2n+1} = (-1)^{n+1} + F_{n+1}L_n\,.$$

From Theorem 1 of [1, p. 39] we have that $s(m)$ is even if $m > 2$.

An equivalent form of the following theorem can be found in Vinson [1, p. 42].

Theorem 1. We have

i) $s(m) = 4f(m)$ if and only if $m > 2$ and $f(m)$ is odd.

ii) $s(m) = f(m)$ if and only if $m = 1$ or 2 and $s(m)/2$ is odd.

iii) $s(m) = 2f(m)$ if and only if $f(m)$ is even and $s(m)/2$ is even.

To prove the above theorem it is sufficient, in view of Theorem 3 by Vinson [1, p. 42], to prove the following:

Lemma. $m = 1$ or 2 or $s(m)/2$ is odd if and only if $8 \nmid m$ and $2\,|\,f(p)$ but $4 \nmid f(p)$ for every odd prime, p, which divides m.

Proof. Let $m = 1$ or 2 or $s(m)/2$ be odd. If $m = 1$ or 2, then the conclusion is clear. So we may assume that $m > 2$ and $s(m)/2$ is odd. Suppose $8\,|\,m$. Then by (2), $12 = s(8)\,|\,s(m)$. Therefore $s(m)/2$ is even, a contradiction. Hence $8 \nmid m$.

Let p be any odd prime which divides m. From [1, p. 37] and (2), $f(p)\,|\,s(p)\,|\,s(m)$. Therefore $4 \nmid f(p)$. Suppose $2 \nmid f(p)$. Then by Theorem 1 of [1, p. 39] and (2), we have $4f(p) = s(p)\,|\,s(m)$, a contradiction. Thus $2\,|\,f(p)$.

Conversely, let $8 \nmid m$ and $2\,|\,f(p)$ but $4 \nmid f(p)$ for every odd prime, p, which divides m. Let p be any odd prime which divides m and let e be any positive integer. From [1, p. 40] we have that $f(p)$ and $f(p^e)$ are divisible by the same power of 2. Therefore $2\,|\,f(p^e)$ and $4 \nmid f(p^e)$. Then since