$$\frac{a^{-2}}{2} < s(a^2u_2 - u_4) < \frac{a^{-1}}{2}$$

so that if $y = u_4 + s$ then

$$1 - \frac{a^{-1}}{2} < -s(a^2u_2 - y) < 1 - \frac{a^{-2}}{2} .$$

Since

$$1 - \frac{a^{-1}}{2} > 0 \quad \text{and} \quad 1 - \frac{a^{-2}}{2} = \frac{a}{2}$$

it follows that

$$|a^2u_2 - y| < \frac{a}{2} .$$

If there were an integer $w$ such that $|a^2w - y| < ½$ it would follow that

$$a^2|u_2 - w| < \frac{1+a}{2} = \frac{a^2}{2}$$

implying that $w = u_2$ and that $y = u_4$, contradicting the fact that $|y - u_4| = 1$. On the other hand, there is an integer $x = y - u_2$ such that $|ax - y| < ½$ since

$$|ax - y| = |(a-1)y - au_2| = (a-1)|y - a^2u_2| < \frac{a(a-1)}{2} = \frac{1}{2} .$$

The existence of $x$ (and the non-existence of $w$) satisfying these conditions, implies that $y = v_2$ for some $v \in S$. Thus,

$$|a^2u_2 - v_2| < \frac{a}{2} .$$

We now find

$$\begin{aligned}|u_2 - v_0| &= |u_2 + v_1 - v_2| \leqslant |v_2a^{-2} - u_2| + |v_2(1 - a^{-2}) - v_1| \\ &= a^{-2}(|v_2 - a^2u_2| + |v_2a - a^2v_1|) < \frac{a^{-1}}{2} + \frac{a^{-1}}{2} = a^{-1} < 1\end{aligned}$$

so that $u_2 = v_0 \in S_0$.

Combining the results of Lemmas 3, 4, 5 we have

*Theorem.* $S_0 = S_1 \cup S_2$ .

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## A GOLDEN DOUBLE CROSTIC: SOLUTION

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"Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel." J. Kepler. Quotation given in *The Divine Proportion* by Huntley (Dover, New York, 1970, p. 23).

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