MORE ON BENFORD'S LAW

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In a recent note, J. Wlodarski [1] observed that the Fibonacci and Lucas numbers tend to obey Benford's law which states: the probability that a random decimal begins with the digit p is

$$\log_{10}(p+1) - \log_{10} p.$$

(By begins, one means has extreme left digit.)

Wlodarski based his observations on the first 100 Fibonacci and Lucas numbers.

This is a report of a further investigation of the Benford phenomena. In this effort, the first 2000 representatives of both the Fibonacci and Lucas numbers were calculated and examined. The occurrences of the first digits were noted and tabulated. Further this was done for each base $b = 3$ to $b = 10$. The results of these calculations suggest an extended Benford law:

The probability that a random decimal written base b begins with p is

$$\log_{10} \frac{p+1}{p} \cdot \frac{1}{\log_{10} b} = \log_b \frac{p+1}{p} . \tag{1}$$

This result is anticipated by Flehinger [2] and is verified here.

In order to provide the statistical data concerning the Fibonacci and Lucas numbers of large magnitude and to various bases, a computer program was developed. It was written in FORTRAN-IV and has been run on an IBM 360-40. The program can develop the numbers up to $n = 5000$ base 10 using the 1000 digits provided. However, more digits would be needed for a lesser base. As a compromise $n = 2000$ was selected. The proportions of first digits to the various bases is recorded in Tables 1 and 2. Table 3 gives the corresponding results from (1) for comparison.

Table 1
Proportion of First Digits of Lucas Numbers

	Digits								
Base	1	2	3	4	5	6	7	8	9
10	.30100	.17600	.12550	.09650	.07950	.06650	.05850	.05100	.04500
9	.31800	.18150	.13300	.10250	.08300	.07000	.05900	.05300	
8	.33350	.19450	.13950	.10600	.08850	.07400	.06400		
7	.35450	.20850	.15000	.11300	.09350	.08050			
6	.37800	.22400	.16150	.12500	.10250				
5	.43050	.25100	.17950	.13900					
4	.50100	.29150	.20750						
3	.63650	.36350							

Table 2
Proportion of First Digits of Fibonacci Numbers

	Digits								
Base	1	2	3	4	5	6	7	8	9
10	.30050	.17650	.12500	.09650	.07950	.06650	.05750	.05200	.04600
9	.31400	.18650	.13200	.09900	.08300	.06950	.06200	.05400	
8	.33400	.19500	.13900	.10600	.08800	.07350	.06450		
7	.35750	.20900	.14600	.11550	.09200	.08000			
6	.38600	.22800	.16050	.12400	.10150				
5	.43100	.25250	.17800	.13850					
4	.49950	.29200	.20850						
3	.62800	.37200							

Table 3
Values of $\log_b (n+1)/n$

	n								
Base	1	2	3	4	5	6	7	8	9
10	.30103	.17609	.12494	.09691	.07918	.06695	.06099	.04815	.04576
9	.31547	.18453	.13093	.10156	.08298	.07016	.06391	.05046	
8	.33223	.19434	.13789	.10695	.08739	.07389	.06731		
7	.35621	.20837	.14784	.11467	.09369	.07922			
6	.38685	.22629	.16056	.12454	.10175				
5	.43068	.25193	.17875	.13865					
4	.50000	.29248	.20752						
3	.63093	.36907							

REFERENCES

1. J. Wlodarski, "Fibonacci and Lucas Numbers Tend to Obey Benford's Law," *The Fibonacci Quarterly,* February 1971, Vol. 9, No. 1, pp. 87–88.
2. B. J. Flehinger, "On the Probability that a Random Integer has Initial Digit A," *Amer. Math. Monthly,* Vol. 73 (1966), pp. 1056–1061.

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