A PRIMER FOR THE FIBONACCI NUMBERS, PART XVI
THE CENTRAL COLUMN SEQUENCE

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1. INTRODUCTION

The rows of Pascal's triangle with *even* subscripts have a middle term

$$A_n = \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

since

$$\binom{n}{k} = \binom{n}{n-k},$$

for $0 \le k \le n$. We shall now derive the generating function

$$A(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n .$$

From

$$A_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

one easily gets

$$(n+1)A_{n+1} = 2(2n+1)A_n .$$

2. GENERATING FUNCTION

From

$$A(x) = \sum_{n=0}^{\infty} A_n x^n = A_0 + \sum_{n=0}^{\infty} A_{n+1} x^{n+1}$$

so that by differention

$$xA'(x) = x \sum_{n=0}^{\infty} (n+1)A_{n+1} x^n = \sum_{n=0}^{\infty} nA_n x^n .$$

From tne relation

$$(n+1)A_{n+1} = 2(2n+1)A_n$$

then

$$A'(x) = \sum_{n=0}^{\infty} (n+1)A_{n+1}x^n = \sum_{n=0}^{\infty} 2(2n+1)A_n x^n = 2\left(\sum_{n=0}^{\infty} 2(nA_n)x^n + \sum_{n=0}^{\infty} A_n x^n \right)$$

so that

$$A'(x) = 2(2xA'(x) + A(x)).$$

Solving for $A'(x)$, one gets, upon dividing by $A(x)$,

$$\frac{A'(x)}{A(x)} = \frac{2}{(1-4x)}$$

from which it follows that

$$\ln A(x) = -\tfrac{1}{2}\ln(1-4x) + \ln C.$$

Thus

$$A(x) = \frac{C}{\sqrt{1-4x}}\,,$$

but $A_0 = A(0) = 1$ implies $C = 1$, so that

$$A(x) = \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} A_n x^n.$$

3. CATALAN NUMBERS

Suppose you know that the Catalan numbers have the form

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad C_0 = 0,$$

and wish to derive the generating function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n.$$

Recall that

$$A(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}\,.$$

Then

$$C(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n = \sum_{n=0}^{\infty}\frac{1}{n+1}A_n x^n.$$

Thus, if we integrate the series for $A(x)$, term-by-term,

$$\int\frac{dx}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\frac{1}{n+1}A_n x^{n+1} + C^*.$$

But

$$\int\frac{dx}{\sqrt{1-4x}} = -\tfrac{1}{2}\sqrt{1-4x} = xC(x) + C^*,$$

which implies $C^* = -\tfrac{1}{2}$. This can be solved for

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

We now show how to derive the central sequence for the trinomial triangle.

4. THE TRINOMIAL TRIANGLE – CENTRAL TERM

Consider the triangular array

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & 1 & & & & \\
1 & 2 & 3 & 2 & 1 & & \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & x & y & z & & & \\
 & & & w & & &
\end{array}$$

where $w = x + y + z$ shows the relation between the elements of the array. It is induced by the expansion of

$$(1+x+x^2)^n, \qquad n = 0, 1, 2, 3, \cdots .$$

Let

$$(1+x+x^2)^n = \sum_{m=0}^{2n} \beta_m x^m = \sum_{k=0}^{n} \binom{n}{k} x^{2k}(1+x)^{n-k}$$

$$= \binom{n}{0}(1+x)^n + \binom{n}{1}(1+x)^{n-1}x^2 + \cdots + \binom{n}{k}(1+x)^{n-k}x^{2k} + \cdots .$$

The coefficient β_n is the central term and is given by

$$\beta_n = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n-1}{n-2} + \cdots + \binom{n}{a}\binom{n-a}{n-2a} ,$$

where $a = [n/2]$. The β_n may be written in several forms.

$$\beta_n = \sum_{k=0}^{[n/2]} \binom{n}{k}\binom{n-k}{n-2k} = \sum_{k=0}^{[n/2]} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} ,$$

since

$$\binom{n}{k}\binom{n-k}{k} = \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{k!(n-2k)!} = \frac{n!\,(2k)!}{(2k)!(n-2k)!k!k!} = \binom{n}{2k}\binom{2k}{k} .$$

We now derive the central term generating function,

$$B(x) = \frac{1}{\sqrt{1-2x-3x^2}} = \sum_{n=0}^{\infty} \beta_n x^n .$$

Thus

$$B(x) = \sum_{m=0}^{\infty} \beta_m x^m = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{[m/2]} \binom{m}{2k}\binom{2k}{k} \right) x^m = \sum_{k=0}^{\infty} \left(\sum_{m=2k}^{\infty} \binom{m}{2k} x^m \right) \binom{2k}{k},$$

since

$$\binom{m}{2k} = 0 \quad \text{if} \quad 0 \leqslant m < 2k.$$

Thus

$$= \sum_{k=0}^{\infty} \binom{2k}{k} \sum_{m=0}^{\infty} \binom{m}{2k} x^m = \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{x^{2k}}{(1-x)^{2k+1}} \right) ,$$

since

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n .$$

But

$$A(x) = \sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

so that

$$B(x) = \frac{1}{1-x}\, A\left(\frac{x^2}{(1-x)^2} \right) = \frac{1}{1-x}\, \frac{1}{\sqrt{1-4\left(\frac{x}{1-x}\right)^2}} = \frac{1}{\sqrt{1-2x-3x^2}} .$$

This completes the derivation.

Thus, $A(x)$, the generating function for central term of the evenly subscripted rows of Pascal's triangle, is related by a transformation to the central term generating function for the trinomial triangle

$$B(x) = \frac{1}{1-x}\,A\left(\frac{x^2}{(1-x)^2}\right).$$

Much more can be done with this but that is another paper and is covered in Rondeau [2] and Anaya [1]. It should be noted that the generating function $B(x)$ could also have been derived by Lagrange's Theorem as in [6].

Sequence 456, in [4], is the Catalan sequence for the Trinomial Triangle $1, 1, 2, 4, 9, 21, \cdots, C_n^*, \cdots$. This sequence C_n^* can be obtained from the regular Catalan $1, 1, 2, 5, 14, 42, \cdots$ if we truncate the first term, by repeated differencing. (See [1].)

Trinomial Catalan Nos.

$$\begin{array}{lccccccl}
 & 1 & 2 & 5 & 14 & 42 & \cdots & C_n \text{ Catalan Numbers} \\
\Delta C_n & & 1 & 3 & 9 & 28 & & \\
\Delta^2 C_n & & & 2 & 6 & 19 & & \\
\Delta^3 C_n & & & & 4 & 13 & & \\
\Delta^4 C_n & & & & & 9 & &
\end{array}$$

The Catalan generating function $C(x)$ is

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

$$C^2(x) = \frac{C(x)-1}{x} = \sum_{n=0}^{\infty} C_{n+1}x^n = \frac{1-2x-\sqrt{1-4x}}{2x^2}$$

Let $C_n^*(x)$ be the generating function for Catalan numbers for the Trinomial Triangle. This is

$$C^*(x) = \frac{1}{1+x}\,C^2\left(\frac{x}{1+x}\right) = \frac{1}{1+x}\left[\frac{1-\dfrac{2x}{1+x}-\sqrt{1-\dfrac{4x}{1+x}}}{2\,\dfrac{x^2}{(1+x)^2}}\right]$$

$$= \frac{1-x-\sqrt{(1+x)(1-3x)}}{2x^2} = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

We can also get $C^*(x)$ from regular Catalan number generator by another transformation related to summation Webs

Catalan Numbers →

Trinomial Catalan Numbers ↘

$$\begin{array}{ccccccccc}
1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \\
 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 14 \;\cdots \\
 & & 2 & 2 & 3 & 4 & 7 & 10 & \\
 & & & 4 & 5 & 7 & 11 & 17 & \\
 & & & & 9 & 12 & 18 & & \\
 & & & & & 21 & 30 & & \\
 & & & & & & 51 & &
\end{array}$$

Here the Catalan number generator is

$$C(x^2) = \frac{1-\sqrt{1-4x^2}}{2x^2}$$

$$C^*(x) = \frac{1}{1-x}\,C\left(\left(\frac{x}{1-x}\right)^2\right) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

This is the same transformation we saw earlier.

5. $B(x)$ FROM THE DIFFERENCE EQUATION

In Riordan [3, p. 74], they give the recurrence for the numbers β_n, the central terms in the rows of a trinomial triangle. This is

$$n\beta_n = (2n-1)\beta_{n-1} + 3(n-1)\beta_{n-2}.$$

We shall now derive this.

We will start with the well known generating function for the Legendre Polynomials

$$\frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} P_n(t)x^n.$$

We introduce a phantom parameter t in the generating function for $B(x)$.

$$B(x,t) = \frac{1}{\sqrt{1-2xt-3x^2}} = \sum_{n=0}^{\infty} M_n(t)x^n,$$

where clearly $B(x,1) = B(x)$ and $M_n(1) = \beta_n$.

Let

$$x_1 = -i\sqrt{3}x \quad \text{and} \quad t_1 = \frac{it}{\sqrt{3}},$$

then

$$\sum_{n=0}^{\infty} M_n(t)x^n = \frac{1}{\sqrt{1-2xt-3x^2}} = \frac{1}{\sqrt{1-2x_1t_1+x_1^2}} = \sum_{n=0}^{\infty} P_n(t_1)x_1^n.$$

$$= \sum_{n=0}^{\infty} P_n\left(\frac{it}{\sqrt{3}}\right)(-i\sqrt{3}x)^n.$$

We note $M_n(1) = \beta_n$, then

$$\beta_n = (-i\sqrt{3})^n P_n(i/\sqrt{3}).$$

The Legendre Polynomials obey the recurrence relation

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

for $n \geqslant 0$, with $P_0(x) = 1$ and $P_1(x) = x$. From $P_0(x) = 1$, then

$$\beta_0 = (-i\sqrt{3})^0 P_0(i/\sqrt{3}) = 1$$

and from $P_1(x) = x$, then $\beta_1 = (-i\sqrt{3})(i/\sqrt{3}) = 1$. Thus directly substituting $P_n(x)$, with $x = i/\sqrt{3}$ the recurrence relation becomes

$$nP_n(i/\sqrt{3}) = (2n-1)\frac{1}{\sqrt{3}}P_{n-1}(i/\sqrt{3}) - (n-1)P_{n-2}(i/\sqrt{3})$$

and

$$n(-\sqrt{3}i)^n P_n(i/\sqrt{3}) = (2n-1)(-\sqrt{3}i)^n \frac{i}{\sqrt{3}} P_{n-1}(i/\sqrt{3}) - (n-1)(-i\sqrt{3})^n P_{n-2}(i/\sqrt{3}).$$

Since

$$\beta_n = (-i\sqrt{3})^n P_n(i/\sqrt{3}),$$

this yields

$$n\beta_n = (2n-1)\beta_{n-1} + 3(n-1)\beta_n,$$

with $\beta_0 = 1$, $\beta_1 = 1$ as was to be shown.

We note in passing that

$$\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 3.$$

6. FROM THE RECURRENCE TO THE GENERATING FUNCTION

We now go from the recurrence relation

$$(n+2)\beta_{n+2} = (2n+3)\beta_{n+1} + 3(n+1)\beta_n\,,$$

with $\beta_0 = \beta_1 = 1$, back to the generating function.

Let

$$B(x) = \sum_{n=0}^{\infty} \beta_n x^n\,,$$

then

$$xB'(x) = \sum_{n=0}^{\infty} n\beta_n x^n\,,$$

$$3xB'(x) + 3B(x) = \sum_{n=0}^{\infty} 3(n+1)\beta_n x^n\,.$$

Further

$$xB'(x) - 0\cdot\beta_0 - x\beta_1 = \sum_{n=2}^{\infty} n\beta_n x^n$$

or

$$(B'(x)-1)/x = \sum_{n=0}^{\infty} (n+2)\beta_{n+2}x^n\,.$$

Next,

$$B(x) = 1 + x\sum_{n=0}^{\infty} \beta_{n+1}x^n, \quad B'(x) = \sum_{n=0}^{\infty} \beta_{n+1}x^n + \sum_{n=0}^{\infty} n\beta_{n+1}x^n\,,$$

$$\frac{B(x)-1}{x} = \sum_{n=0}^{\infty} \beta_{n+1}x^n, \quad 2B'(x) + \frac{B(x)-1}{x} = \sum_{n=0}^{\infty} (2n+3)\beta_{n+1}x^n\,.$$

Thus, from the recurrence relation, we may write

$$\frac{B'(x)-1}{x} = 2B'(x) + \frac{B(x)-1}{x} + 3xB'(x) + 3B(x)$$

or

$$B'(x)(1-2x-3x^2) = (3x+1)B(x), \quad \frac{B'(x)}{B(x)} = \frac{3x+1}{1-2x-3x^2}\,.$$

Integrating, $\ln B(x) = -\tfrac{1}{2}\ln(1-2x-3x^2) + \ln C$. Thus

$$B(x) = \frac{C}{\sqrt{1-2x-3x^2}}\,,$$

and since $B(0) = \beta_0 = 1$, it follows that $C = 1$. This concludes the discussion.

REFERENCES

1. Janet E. Anaya, "Web Sequences and Multisecting the Column Generating Functions of Pascal's Triangle," Unpublished Master's Thesis, San Jose State University, May, 1977.
2. Michel Rondeau, "The Generating Functions for the Vertical Columns of the $(N+1)$-Nomial Triangles," Unpublished Master's Thesis, San Jose State University, December, 1973.
3. John Riordan, *Combinatorial Identities,* John Wiley and Sons, New York, N.Y., 1968, p. 74.
4. N. J. A. Sloane, *Handbook of Integer Sequences,* The Academic Press, New York, N.Y., 1973.
5. Earl D. Rainville, *Special Functions,* McMillan Co., New York, N. Y., 1960.
6. V. E. Hoggatt, Jr., and Paul S. Bruckman, "H-Convolution Transform," *The Fibonacci Quarterly,* Vol. 13, No. 4, Dec. 1975, pp. 357–368.

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