FIBONACCI TILING AND HYPERBOLAS

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ABSTRACT

A sequence of rectangles R_n is generated by adding squares cyclically to the East, N, W, S side of the previous rectangle. The centers of R_n fall on a certain hyperbola, in a manner reminiscent of multiplication in a real quadratic number field.

INTRODUCTION

We take a special case for simplicity. Suppose R_1 is the square $-1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1$. R_2 is the rectangle $-1 \leqslant x \leqslant 3, -1 \leqslant y \leqslant 1$. R_3 is the rectangle $-1 \leqslant x \leqslant 3, -1 \leqslant y \leqslant 5$. Let F_n denote the n^{th} Fibonacci number. Then R_n has sides $2F_n$ and $2F_{n-1}$ for all n.

We ask for information about the center (x_n, y_n) of R_n. This search leads us to the ring $R \otimes R$ in which $R \otimes R$ is given pointwise addition and multiplication. We close with an examination of "rotations" and linear fractional mappings of $R \otimes R$. Certain classes of hyperbolas remain invariant under such mappings.

1. DEFINITIONS AND STATEMENT OF RESULTS

Let $a, b > 0$. Suppose a sequence of rectangles is generated in the following manner. The initial rectangle has center (0,0) and positive dimensions X_1, Y_1. If the n^{th} rectangle R_n has dimensions X_n, Y_n then R_{n+1} is the union of R_n with an incremental rectangle on the East, N. W, S side of R_n according as $n \equiv 1, 2, 3, 0 \bmod 4$. The dimensions of the incremental rectangle are $aY_n + b$, Y_n if $n \equiv 1 \bmod 2$, and X_n, $aX_n + b$ if $n \equiv 0 \bmod 2$.

Theorem. Let (x_n, y_n) be the center of R_n. Let $D = \tfrac{1}{2}(aY_1 + b)$, $E = \tfrac{1}{2}(aX_1 + b)$.
Then for all $n \geqslant 1$, (x_n, y_n) lies on the right hyperbola

$$H = \left\{ (x,y) : x^2 + axy - y^2 - Dx + Ey = 0 \right\}.$$

Further, if h is the center of H, then the area enclosed by H and the rays

$$\overline{h, (x_n, y_n)} \quad \text{and} \quad \overline{h, (x_{n+4}, y_{n+4})}$$

is independent of n.

REMARK. The proof that the (x_n, y_n) lie on H is a rather ordinary induction. To prove that the areas enclosed by H and rays from adjacent rectangle centers to h are all equal, we introduce the ring $R \otimes R$.

Definition. $R \otimes R$ is the ring $R \otimes R$ with addition $(x,y) + (x', y') = (x + x', y + y')$ and multiplication $(x,y) \cdot (x', y') = (x \cdot x', y \cdot y')$.

Definition. If $(x,y) \in R \times R$, $N(x,y) = xy$; and Arg $(x,y) = \log |(y/x)|$ if $xy \neq 0$.

Definition. If $N(x,y) \neq 0$,

$$\frac{(x', y')}{(x,y)} = \left(\frac{x'}{x}, \frac{y'}{y} \right).$$

REMARK. $N(x,y) = 1$ is the hyperbola $xy = 1$. Arg (x,y) is the area enclosed by $N(x,y) = 1$ and the rays

$$\overline{(0,0), (|x|, |y|)} \quad \text{and} \quad \overline{(0,0), (|y|, |x|)}.$$

It is for this area property, so similar to the one stated in Theorem 1, that we introduce $R \otimes R$.

Theorem 2. Let k be real, $a, b, c, d, z_0 \in R \otimes R$. Assume not both $a,b = (0,0)$ and not both $c_1, d_1 = 0$ and not both $c_2, d_2 = 0$. Let $k \neq 0$. (Here $(c_1, c_2) = c$ and $(d_1, d_2) = d$.)

Let

$$f(z) = \frac{az+b}{cz+d}$$

for all z such that $N(cz+d) \neq 0$. Then the image under f of $\{z : N(z - z_0 = k\}$ is of the form

$$\{w : N(w - w_0) = k'\}$$

where no more than 4 points are missing.

REMARK. Thus except for technicalities, a linear fractional maps hyperbolas of the form $N(z - z_0) = k$ to hyperbolas of the same form. The analogy with the complex numbers, where linear fractionals map circles to circles, suggests many more similar results which space does not permit us to list.

2. PROOFS. THEOREM 1, PART 1

The reader may verify by direct calculation that the first couple of (x_n, y_n) lie on H. We now claim that

$$2x_n + ay_n + \tfrac{1}{2}(aY_n + b) = D \quad \text{if} \quad n \equiv 1 \mod 4.$$

$$-ax_n + 2y_n + \tfrac{1}{2}(aX_n + b) = E \quad \text{if} \quad n \equiv 2 \mod 4.$$

$$2x_n + ay_n - \tfrac{1}{2}(aY_n + b) = D \quad \text{if} \quad n \equiv 3 \mod 4,$$

and

$$-ax_n + 2y_n - \tfrac{1}{2}(aX_n + b) = E \quad \text{if} \quad n \equiv 0 \mod 4.$$

Observe that if $(x_n, y_n) \in H$ and the claim is true for n, then $(x_{n+1}, y_{n+1}) \in H$. Thus we need only prove the claim to show that all (x_n, y_n) are on H.

Proof of claim, $n \equiv 1 \mod 2$.

If the claim is true for some $n \equiv 1 \mod 4$, then

$$2x_n + ay_n + \tfrac{1}{2}(aY_n + b) = D.$$

We show that the claim follows for $n+2$.

For,

$$x_{n+2} = x_n + \tfrac{1}{2}(aY_n + b), \qquad y_{n+2} = y_n + \tfrac{1}{2}(b + aX_n + a^2Y_n + ab),$$

and

$$Y_{n+2} = Y_n + b + aX_n + a^2Y_n + ab.$$

Thus

$$\begin{aligned} 2x_{n+2} + ay_{n+2} - \tfrac{1}{2}(aY_{n+2} + b) &= 2(x_n + \tfrac{1}{2}(aY_n + b)) + a(y_n + \tfrac{1}{2}(b + aX_n + a^2Y_n + ab)) \\ &\quad - \tfrac{1}{2}(b + a(Y_n + b + aX_n + a^2Y_n + ab)) = \\ \text{(by claim)} \quad &= D + (aY_n + b) + (\tfrac{1}{2}ab + \tfrac{1}{2}a^2X_n + \tfrac{1}{2}a^3Y_n + \tfrac{1}{2}a^2b) - \tfrac{1}{2}b \\ &\quad - \tfrac{1}{2}aY_n - \tfrac{1}{2}ab - \tfrac{1}{2}a^2X_n - \tfrac{1}{2}a^3Y_n - \tfrac{1}{2}a^2b \\ &\quad - \tfrac{1}{2}(aY_n + b) = D. \end{aligned}$$

Similarly, if the claim is true for some $n \equiv 2 \mod 4$ it is true for $n+2$, if true for some $n \equiv 3 \mod 4$ it is true for $n+2$, and if true for $n \equiv 0 \mod 4$ it is true for $n+2$. Thus it is only necessary to check that the claim is true for $n = 1$ and $n = 2$. If $n = 1$, x_n and $y_n = 0$ and $\tfrac{1}{2}(aY_1 + b) = D$ by definition. $x_2 = \tfrac{1}{2}(aY_1 + b)$, and $y_2 = 0$. $X_2 = X_1 + aY_1 + b$, and $Y_2 = Y_1$. Thus

$$-ax_2 + 2y_2 + \tfrac{1}{2}(aX_2 + b) = -\tfrac{1}{2}a(aY_1 + b) + \tfrac{1}{2}(aX_1 + a^2Y_1 + a^2Y_1 + ab + b) = \tfrac{1}{2}(aX_1 + b) = E$$

by definition. This proves the claim, and hence the centers of R_n lie on H.

For the second part of Theorem 1, we note that H is a hyperbola whose asymptotes are perpendicular. It is therefore similar, in the geometric sense, to the hyperbola $xy = 1$. Let

$$\varphi : R \otimes R \to R \otimes R$$

be a similarity mapping which takes H onto $xy = 1$.

For each n, the line (x_{n-1}, y_{n-1}), (x_n, y_n) is perpendicular to (x_n, y_n), (x_{n+1}, y_{n+1}). This property is preserved under the similarity mapping of H onto $xy = 1$.

Let $z_n = (x'_n, y'_n) = \varphi(x_n, y_n)$. Let c be the slope of the line from (x'_1, y'_1) to (x'_2, y'_2). Let $C = (c, 1/c)$. $(c \neq 0)$. Then (with the help of a little algebra)

$$\begin{aligned} z_n &= -C^{-n+1}z_1^{-1} && \text{if} \quad n \equiv 2 \bmod 4, \\ &-C^{n-1}z_1 && \text{if} \quad n \equiv 3 \bmod 4 \\ &+C^{-n+1}z_1^{-1} && \text{if} \quad n \equiv \quad \bmod 4 \end{aligned}$$

and

$$z_n = +C^{n-1}z_1 \quad \text{if} \quad n \equiv 1 \bmod 4.$$

Now the region enclosed by the lines from (0,0) to z_n and to z_{n+4}, and by $xy = 1$, has area

$$|\tfrac{1}{2}(\operatorname{Arg}(z_{n+4}) - \operatorname{Arg}(z_n))| = |\tfrac{1}{2}\operatorname{Arg}(z_{n+4}/z_n)| = |\tfrac{1}{2}\operatorname{Arg}(C^4)| \quad \text{or} \quad |\tfrac{1}{2}\operatorname{Arg}(C^{-4})|$$

depending on whether n is odd or even. Either way, since Arg (C) = Arg (C^{-1}), all such regions have equal areas.

Thus the corresponding regions bounded by lines from the center of H to the (x_n, y_n) also have areas equal to each other's, since φ multiplies areas by a constant.

The mapping $r : z \to C^4 z$ of $R \otimes R$ onto $R \otimes R$ may be viewed as a "rotation" of $R \otimes R$, since it changes Arg (z) but not $N(z)$. Clearly r sends hyperbolas of the form $N(z) = k$ into themselves. This is reminiscent of linear fractional transformations of the complex plane. Although there is no direct further bearing on Fibonacci tiling, we are inclined to note some similarities.

Proof of Theorem 2. Fix $a,b,c,d \in R \otimes R$. Let $(c_1, c_2) = c$ and $(d_1, d_2) = d$. Suppose not both a and b = (0,0), and $(c_1, d_1) \neq (0,0)$ $(c_2, d_2) \neq (0,0)$. Fix $x_o, y_o, k \neq 0 \in R$.

Lemma 1. Under the above conditions, there exist $x_1, y_1, x_2, y_2, K \in R$ such that

$$K \neq 0,\ x_1 \neq x_0,\ x_1 \neq -d_1/c_1,\ y_1 \neq y_0,\ y_1 \neq -d_2/c_2,\ x_2 \neq x_1,$$
$$x_2 \neq -d_1/c_1,\ y_2 \neq y_1, y_2 \neq -d_2/c_2,$$

and such that $(x - x_0)(y - y_0) = k$ if and only if $(x - x_1)(y - y_1)/(x - x_2)(y - y_2) = K$ or $(x,y) = (x_1, y_2)$ or (x_2, y_1).

Proof. Select some $k \neq 0, 1$ such that

$$(K-1)(k - x_0y_0) + K^{-2}(K-1)^2x_0y_0 \neq 0.$$

Fix K. Let

$$x_2 = K^{-1}((K-1)x_0 + x_1), \quad y_2 = K^{-1}((K-1)y_0 + y_1).$$

Then the equation

$$k - x_0y_0 = (K-1)^{-1}(x_1y_1 - K^{-2}((K-1)x_0 + x_1)((K-1)y_0 + y_1)$$

has a range of solutions x_1, y_1 in which y_1 is a non-constant continuous function of x_1.

When the above conditions are satisfied, and $x_1 \neq x_0, y_1 \neq y_0$,

$$(x - x_0)(y - y_0) = k \Leftrightarrow (x - x_1)(y - y_1) = K(x - x_2)(y - y_2).$$

Thus Lemma 1.

We may restate this as saying that except for a special class of degenerate hyperbolas, every hyperbola $N(z - z_0) = k$ can be put in the form

$$\frac{N(z - z_1)}{N(z - z_2)} = K.$$

Now let $\lambda \in R \otimes R$,

$$\lambda = \left(\frac{c_1x_2 + d_1}{c_1x_1 + d_1}, \frac{c_2y_2 + d_2}{c_2y_1 + d_2}\right).$$

Let $w_1 = f(z_1)$, $w_2 = f(z_2)$. Then

$$\frac{w - w_1}{w - w_2} = \lambda \frac{z - z_1}{z - z_2} \Leftrightarrow w = f(z) \quad \text{or} \quad w = w_2,\ z = z_2\ .$$

Thus

$$\frac{N(z-z_1)}{N(z-z_2)} = K$$

has image

$$\frac{N(w-w_1)}{N(w-w_2)} = KN(\lambda).$$

By our previous results this is also a hyperbola of the same sort.

REMARK. Thus except for isolated points for which necessary divisions are impossible in $R \otimes R$, $R \otimes R$ behaves just like $\mathbb{C}$ with respect to linear fractional mappings.

One could show without great difficulty that the maps f of Theorem 2, are "conformal," in the $R \otimes R$ sense. Self mappings of the "unit circle" $N(z) \leqslant 1$ have properties analogous to their counterparts over $\mathbb{C}$. But the prospects along this line are quite limited. $R \otimes R$ is only a curiosity, and cannot (in my opinion) support a deep and rich theory.

For those familiar with the number theory of $Q(\sqrt{5})$, we remark that for the example of the introduction, by embedding $Q(\sqrt{5})$ in $R \otimes R$ one may show that the (x_n, y_n) consist of all the integer points on

$$x^2 + xy - y^2 - x + y = 0,$$

except for (0,1).

REFERENCES

1. James Hafner, "Generalized Fibonacci Tiling Made Easy," to appear.
2. Herbert Holden, "Fibonacci Tiles," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), p. 45.

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