DIAGONAL FUNCTIONS

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INTRODUCTION

The object of this article is to combine and generalize some of the ideas in [1] and [2] which dealt with extensions to the results of Jaiswal, and of Hansen and Serkland. [See [1] and [2] for the references.]

We commence with the pair of sequences $\{A_n\}$ and $\{B_n\}$ for which

(1) $$A_{n+2} = xA_{n+1} + A_n, \qquad A_0 = 0, \quad A_1 = 1 \quad (x \neq 0)$$

(2) $$B_{n+2} = xB_{n+1} + B_n, \qquad B_0 = 2, \quad B_1 = x$$

with the special properties

(3) $$A_{n+1} + A_{n-1} = B_n$$

(4) $$B_{n+1} + B_{n-1} = (x^2+4)A_n\ .$$

[See [2], where c has been replaced by x.]

The first few terms of these sequences $\{A_n\}$ and $\{B_n\}$ are

(5) $$\begin{array}{l} 0 \\ 1 \\ x \\ x^2+1 \\ x^3+2x \\ x^4+3x^2+1 \\ x^5+4x^3+3x \\ x^6+5x^4+6x^2+1 \\ x^7+6x^5+10x^3+4x \\ x^8+7x^6+15x^4+10x^2+1 \\ \hline \end{array}$$

(6) $$\begin{array}{l} 2 \\ x \\ x^2+2 \\ x^3+3x \\ x^4+4x^2+2 \\ x^5+5x^3+5x \\ x^6+6x^4+9x^2+2 \\ x^7+7x^5+14x^2+7x \\ x^8+8x^6+20x^4+16x^2+2 \\ x^9+9x^7+27x^5+30x^3+9x \\ \hline \end{array}$$

RISING DIAGONAL FUNCTIONS

Consider the *rising diagonal functions* of x, $R_i(x)$, $r_i(x)$ for (5) and (6), respectively (indicated by unbroken lines):

(7) $$\begin{cases} R_1(x) = 1 & R_2(x) = x \qquad R_3(x) = x^2 \qquad R_4(x) = x^3+1 \\ R_5(x) = x^4+2x & R_6(x) = x^5+3x^2 \qquad R_7(x) = x^6+4x^3+1 \qquad R_8(x) = x^7+5x^4+3x \\ R_9(x) = x^8+6x^5+6x^2 & R_{10}(x) = x^9+7x^6+10x^3+1, \cdots \end{cases}$$

(8) $$\begin{cases} r_1(x) = 2 & r_2(x) = x \qquad r_3(x) = x^2 \qquad r_4(x) = x^3+2 \\ r_5(x) = x^4+3x & r_6(x) = x^5+4x^2 \qquad r_7(x) = x^6+5x^3+2 \qquad r_8(x) = x^7+6x^4+5x \\ r_9(x) = x^8+7x^5+9x^2 & r_{10}(x) = x^9+8x^6+14x^3+2, \cdots \end{cases}$$

Define

(9) $$R_0(x) = r_0(x) = 0.$$

Observe that, in (7), (8) and (9), for $n \geqslant 3$,

$$\text{(10)} \qquad \begin{cases} r_n(x) = R_n(x) + R_{n-3}(x) \\ R_n(x) = xR_{n-1}(x) + R_{n-3}(x) \\ r_n(x) = xr_{n-1}(x) + r_{n-3}(x) \end{cases} .$$

Generating functions for the rising diagonal polynomials are

$$\text{(11)} \qquad A \equiv A(x,t) \equiv (1 - xt - t^3)^{-1} = \sum_{n=1}^{\infty} R_n(x)t^{n-1}$$

and

$$\text{(12)} \qquad B \equiv B(x,t) \equiv (1+t^3)(1 - xt - t^3)^{-1} = \sum_{n=2}^{\infty} r_n(x)t^{n-1} .$$

Calculations with (11) and (12) yield the partial differential equations

$$\text{(13)} \qquad t\,\frac{\partial A}{\partial t} - (x + 3t^2)\,\frac{\partial A}{\partial x} = 0$$

and

$$\text{(14)} \qquad t\,\frac{\partial B}{\partial t} - (x + 3t^2)\,\frac{\partial B}{\partial x} - 3B + 3A = 0.$$

leading to

$$\text{(15)} \qquad xR'_{n+2}(x) + 3R'_n(x) - (n+1)R_{n+2}(x) = 0$$

$$\text{(16)} \qquad xr'_{n+2}(x) + 3r'_n(x) - (n-2)r_{n+2}(x) - 3R_{n+2}(x) = 0 \quad (n \geqslant 2),$$

where the prime denotes differentiation with respect to x.

Comparing coefficients of t^n in (11) we deduce that

$$\text{(17)} \qquad R_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i} \quad (n \geqslant 3)$$

where $[n/3]$ is the integral part of $n/3$.

Similarly, from (12) we derive

$$\text{(18)} \qquad r_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i} + \sum_{i=0}^{[(n-3)/3]} \binom{n-3-2i}{i} x^{n-3i} \quad (n \geqslant 3)$$

as may also be readily seen from the first statement in (10).

Simple examples of rising diagonal sequences are:

(a) for the Fibonacci and Lucas sequences $(x = 1)$:

(19) 0 1 1 1 2 3 4 6 9 13 19 ⋯

(20) 2 1 1 3 4 5 8 12 17 25 ⋯

and

(b) for the Pell sequences $(x = 2)$:

(21) 0 1 2 4 9 20 44 97 214 ⋯

(22) 2 2 4 10 22 48 106 234 ⋯

DESCENDING DIAGONAL FUNCTIONS

From (5) and (6), the descending diagonal functions of x, $D_i(x)$, $d_i(x)$ (indicated by broken lines) are:

$$\text{(23)} \qquad \begin{cases} D_1(x) = 1 & D_2(x) = x+1 & D_3(x) = (x+1)^2 & D_4(x) = (x+1)^3 \\ D_5(x) = (x+1)^4 & D_6(x) = (x+1)^5 & D_7(x) = (x+1)^6 & D_8(x) = (x+1)^7, \quad \cdots \end{cases}$$

$$(24)\begin{cases} d_1(x) = 2 \qquad d_2(x) = (x+1)+(x+1)^0 = (x+2)(x+1)^0 = x+2 \\ d_3(x) = (x+1)^2+(x+1) = (x+2)(x+1) \qquad d_4(x) = (x+1)^3+(x+1)^2 = (x+2)(x+1)^2 \\ d_5(x) = (x+1)^4+(x+1)^3 = (x+2)(x+1)^3 \qquad d_6(x) = (x+1)^5+(x+1)^4 = (x+2)(x+1)^4, \end{cases}$$

Define

$$D_0(x) = d_0(x) = 0. \tag{25}$$

Obviously $(n \geqslant 2)$

$$(26)\quad \begin{cases} D_n = (x+1)D_{n-1} = (x+1)^{n-1} \\ d_n = D_n + D_{n-1} = (x+2)D_{n-1} = (x+2)(x+1)^{n-2} \\ d_n = (x+1)d_{n-1} \quad (n > 2) \\ \dfrac{D_n}{D_{n-1}} = \dfrac{d_n}{d_{n-1}} \; (= x+1) \quad (n > 2) \\ \dfrac{D_n}{d_n} = \dfrac{x+1}{x+2}, \end{cases}$$

where, for visual ease, we have temporarily written $D_n \equiv D_n(x)$ and $d_n \equiv d_n(x)$.

Generating functions for the descending diagonal polynomials are

$$A \equiv A(x,t) = [1-(x+1)t]^{-1} = \sum_{n=1}^{\infty} D_n(x)t^{n-1} \tag{27}$$

and

$$B \equiv B(x,t) = (x+2)[1-(x+1)t]^{-1} = \sum_{n=1}^{\infty} d_{n+1}(x)t^{n-1} \tag{28}$$

from which are obtained the partial differential equations

$$t\frac{\partial A}{\partial t} - (x+1)\frac{\partial A}{\partial x} = 0 \tag{29}$$

$$t\frac{\partial B}{\partial t} - (x+1)\frac{\partial B}{\partial x} + (x+1)A = 0, \tag{30}$$

leading to

$$(x+1)D_n'(x) = (n-1)D_n(x) \tag{31}$$

$$(x+1)d_{n+2}'(x) - (n+1)d_{n+2}(x) + (x+1)D_n(x) = 0. \tag{32}$$

Descending diagonal sequences for some well known sequences are:

(a) for the Fibonacci and Lucas sequences $(x = 1)$:

(33)	1	2	4	8	16	32	64	128	$\cdots$	2^n	$\cdots$
(34)	2	3	6	12	24	48	96	192	$\cdots$	$3 \cdot 2^{n-1}$	$\cdots$

and

(b) for the Pell sequences $(x = 2)$:

(35)	1	3	9	27	81	243	729	2187	$\cdots$	3^n	$\cdots$
(36)	2	4	12	36	108	324	972	2916	$\cdots$	$4 \cdot 3^{n-1}$	$\cdots$

CONCLUDING COMMENTS

1. The above results proceed only as far as corresponding work in [1] and [2]. Undoubtedly, more work remains to be done on functions R_i, r_i, D_i, d_i.
2. Excluded from our consideration in this article are the pair of Fermat sequences and the pair of Chebyshev sequences for both of which the criteria (1) and (2) do not hold. [See [2].]

3. Jaiswal, and the author [1], deal only with the rising diagonal functions of Chebyshev polynomials of the first and second kinds.
4. Our special criteria (3) and (4) prevent the use of the more general sequences $\{U_n\}$, $\{V_n\}$ for which

$$U_{n+2} = xU_{n+1} + yU_n \qquad U_0 = 0, \quad U_1 = 1 \quad (x \neq 0,\ y \neq 0)$$
$$V_{n+2} = xV_{n+1} + yV_n \qquad V_0 = 2, \quad V_1 = x\ .$$

See [2] and Lucas [3] pp. 312–313.

5. Finally, in passing, we note that the Pell sequence obtained from (1) with $x = 2$, namely, the sequence 1,2,5,12,29,70, ···, arises from rising diagonals in the "arithmetical square" of Delannoy [Lucas [3] p. 174]

Can any reader inform me, along with a suitable reference, whether Delannoy's "arithmetical square" has been generalized?

REFERENCES

1. A. F. Horadam, "Polynomials Associated with Chebyshev Polynomials of the First Kind," *The Fibonacci Quarterly,*
2. A. F. Horadam, "Generating Identities for Generalized Fibonacci and Lucas Triples," *The Fibonacci Quarterly,*
3. E. Lucas, *Théorie des Nombres,* Gauthier-Villars et Fils, Paris, 1891.

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