ON GENERALIZED $G_{j,k}$ NUMBERS

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Most of this paper was finished prior to the author's involvement in other work [9, 10]. It is the purpose of this exegesis to find a self-contained definition of $\{G_j\}$ which is not dependent on other sequences. Such are (10), (12) and (16). I have defined these numbers in [2, (3)] and [3, (9)]. G numbers of the j^{th} order are:

(1) $$G_{j,k} = 1 + P^*_{j,k} + P_{j,2k-1},$$

where the Lucas complement is by definition

(2) $$P^*_{j,k} = P_{j,k+1} + P_{j,k-1},$$

and where coprime sequences are by definition, j an integer,

(3) $$P_{j,k+1} = jP_{j,k} + P_{j,k-1},$$

and where the initial conditions (IC) are by choice

(3a) $$P_{j,0} = 0 \quad \text{and} \quad P_{j,1} = 1 \quad \text{for all } j.$$

To begin we need the following easily proven identities. The Lucas complement of the Lucas complement is

(4) $$P^*_{j,k+1} + P^*_{j,k-1} = P_{j,k+2} + 2P_{j,k} + P_{j,k-2} = (4+j^2)P_{j,k}.$$

Secondly given any two point recurrence $P_{n+1} = aP_n + bP_{n-1}$ the recurrence among its bisection is known to be

(5) $$P_{n+2} = (a^2+2b)P_n - b^2P_{n-2}.$$

Thirdly we need the central difference operator

(6) $$\delta^2 P_n = (\Delta - \nabla)P_n = P_{n+1} - 2P_n + P_{n-1}$$

and fourthly I define a new operator small psi

(7) $$\psi_j(P_n) = [\delta^2 - j^2]P_n,$$

where j^2 is really j^2 times the identity operator. Note that if $B_{j,n}$ is any generalized bisected coprime sequence with any $B_{j,0}$ and $B_{j,1}$ whatsoever that ψ_j then acts as a null operator, to wit

(8) $$\psi_j(B_{j,n}) = 0 \quad \text{for all } j.$$

Now when $j = 1$ then (7) reduces to $\psi(F_n) = [\delta^2 - 1]F_n$. Consider

(9) $$\psi_j(G_{j,k}) = \psi_j(P^*_{j,k}) - j^2$$

which is obvious from (1) and (8) and the fact that $\psi_j(1) = -j^2$. In (9) elimination of δ^2 via (6) gives

(9a) $$\psi_j(G_{j,k}) = (4+j^2)P_{j,k} - (2+j^2)P^*_{j,k} - j^2.$$

Theorem. The recurrence for $\psi_j(G_{j,k})$ is Fibonacci but for the additive constant j^3.

Proof. Rewrite (9a) as $\psi_jG_{j,k+1}$ and substitute (3) giving

(10) $$\begin{aligned}\psi_j(G_{j,k+1}) &= [j^2+4][jP_{j,k} + P_{j,k-1}] - [j^2+2][jP^*_{j,k} + P^*_{j,k-1}] - j^2\\ &= j\psi_j(G_{j,k}) + \psi_j(G_{j,k-1}) + j^3\end{aligned}$$

Eliminating j^3 by calculating $\psi G_{j,k+1} - \psi G_{j,k}$ obtains

Corollary 1. $$\psi G_{j,k+1} = (j+1)\psi G_{j,k} - (j-1)\psi G_{j,k-1} - \psi G_{j,k-2}.$$

Inserting (7), the definition of psi, one finds the general recurrence

(11) $$G_{j,k+1} = (j^2+j+3)G_{j,k} - (j^3+j^2+3j+2)G_{j,k-1} + (j^3-j^2+3j-2)G_{j,k-2} + (j^2-j+3)G_{j,k-3} - G_{j,k-4}.$$

This recurrence is not messy but instead factors into the crowning equation of this paper

(12) $$(E^2-(j^2+2)E+I)(E^2-jE-I)(E-I)G_{j,k} = 0,$$

where E is the forward shift operator. Note that the first, second and third parentheses of (12) are, in fact, the recurrences for bisected coprime, coprime and constant sequences respectively! A more useful expression in terms of forward and backward difference operators is

(13) $$(\delta^2-I)(\Delta+\nabla-I)\Delta G_{j,k} = 0 = (\Delta^3-2\Delta^2+\Delta-\nabla\delta^2)G_{j,k}$$

only if $j = 1$. Now (12) is more general than (1) and (13) is more general than $\{G_1\} = \cdots 79, 42, 10, 9, 2, 4, 3, 6, 10, 21, 46, 108, \cdots$. An example of (13) is the sequence

(13a) 0, 0, 0, 0, 1, 5, 18, 56, 162, 450, 1221, 3267, 8668, 22880, ⋯, 60204, 158108, 414729,

whose falling diagonal, Δ^t, from the first zero is

(13b) 0, 0, 0, 0, 1, 0, 3, 0, 8, 0, 21, 0, ⋯.

Hence to obtain j^{th} order G numbers some IC must be introduced. First some simplifications. When $j = 1$, then Eqs. (9a), (10) and (11) become

(9b) $$(\delta^2-I)G_k = 5F_k - 3L_k - 1 = -(1+2L_{k-2})$$

(10a) $$\psi G_{k+1} = \psi G_k + \psi G_{k-1} + 1$$

(11a) $$G_{k+1} = 5G_k - 7G_{k-1} + G_{k-2} + 3G_{k-3} - G_{k-4},$$

respectively. Note that (13a) was calculated by (13) and checked by (11a). Also note that (11), (12), (13), (11a) are fifth-degree recurrences. Gould [5] found (11a) independently. Directly from (10) one can find the modified recurrence

(14) $$G_{j,k+1} = (j^2+j+2)G_{j,k} - (j^3+2j)G_{j,k-1} - (j^2-j+2)G_{j,k-2} + G_{j,k-3} + j^3,$$

which, when $j = 1$, becomes

(14a) $$G_{k+1} = 4G_k - 3G_{k-1} - 2G_{k-2} + G_{k-3} + 1$$

and from this latter it is easy to derive the exquisite

(14b) $$\delta^4 G_{k+2} = 3\delta^2 G_{k+1} - G_k + 1.$$

At this point the reader should study Tables 1 and 2. Now a curious fact results from Corollary 1 which I rewrite as

Corollary 1. $$\psi(G_{j,k+1} + G_{j,k-2}) = (1+j)\psi G_{j,k} + (1-j)\psi G_{j,k-1}$$

This says that making both j and k negative reproduces the same recurrence. To be specific replace j by $-j$ and let $n = (1-k)$ and the Corollary regenerates itself. Thus 4, 3, 6, 10, 21, 46, ⋯ has the same recurrence as 4, 4, 9, 18, 42, 101, ⋯. See Table 1.

Lemma. The zeroth term of all $\{G_j\}$ equals the constant 4.

The proof is direct from Eqs. (1) through (3a). Omitting the subscript j for simplicity and recalling that $P_{j,1} = 1$ for all j we have:

$$G_0 = 1 + P_0^* + P_{-1} = 1 + P_1 + P_{-1} + P_{-1} = 1 + 3P_1 = 4$$

(15) $$G_{j,0} = 4.$$

From (12) of paper [3] one may easily find

(16a,b) $$G_{j,1} = (j+2) \quad \text{and} \quad \delta^2 G_{j,0} = G_{j,1}\Delta G_{j,0}$$

Table 1

Array of $G_{j,k}$ Numbers

j/k	−4	−3	−2	−1	0	1	2	3	4	5	6
6	2027452	53120	1444	32	4	8	76	1640	54796	2034896	
5	510354	18761	729	22	4	7	54	843	19629	513402	
4	98532	5392	324	14	4	6	36	382	5796	99574	
3	13090	1154	121	8	4	5	22	146	1309	13364	
2	1020	156	36	4	4	4	12	44	204	1068	
1	42	10	9	2	4	3	6	10	21	46	108
0	4	2	4	2	4	2	4	2	4	2	4
−1	42	18	9	4	4	1	6	2	21	24	108
−2	1020	184	36	8	4	0	12	16	204	804	
−3	13090	1226	121	14	4	−1	22	74	1309	12578	

Table 2

The Table of Differences of G_k

9··· 2 4 3 6 10 21 46 108 ···
−7 2 −1 3 4 11 25 62
9 −3 4 1 7 14 37
−12 7 −3 6 7 23
19 −10 9 1 16
46 −29 19 −8 15
−75 48 −27 23 2
−200 123 −75 −21
323

leaving a fourth initial condition to be chosen in order to define $G_{j,k}$. We may now take this to be

(16c) $$\delta^2 G_{j,1} = 2G_{j,-1}\,.$$

One can also show from (1) or from (12) of paper [3] that

(17) $$G_{j,-2} = (j^2+2)^2 \quad \text{and} \quad G_{j,-1} = j(j-1)+2 = G_{-j+1,-1}$$

for all integer j. At this point it will help the reader to go through an example such as the $j = 3$ case beginning with $P_{3,k} = \cdots 0, 1, 3, 10, 33, 109, 360, 1189, 3927, \cdots$. In fact relations stronger than Corollary 1 exist as is evident from Table 1 where we see that

(18) $$G_{j,k} + G_{j,-k} = G_{-j,k} + G_{-j,-k}$$

for all integer j and k and indeed a special case follows if e is even

(19) $$G_{j,e} = G_{-j,e}$$

Now (18) and (19) are easily proven from (1) and the odd/even properties of F and L sequences.

DIVISIBILITY PROPERTIES

For the study of divisibility properties we are able to rewrite (1) by substituting (6) of [3],

$$P_{2n-1} = P_n^* P_{n-1} - \cos(\pi n)\,,$$

into it giving

(20) $$G_{j,k} = P_{j,k}^*(1+P_{j,k-1}) + 1 + (-1)^{k+1}$$

(20a) $$G_k = L_k(1+F_{k-1}) + 1 + (-1)^{k+1}\,.$$

Hence the divisibility properties of the even G_k are known since Jarden [4, p. 97] has tabulated the divisors of $(1 + F_n)$. The divisibility of the odd G_k is involved. Three divides G_k at intervals of eight starting with

$$k = \cdots -7, 1, 9, 17, 25, 33, \cdots$$

and five divides G_k at intervals of twenty starting with $k = \cdots -3, 17, 37, \cdots$ and proceeding in both directions. Divisibility properties are left for a later paper.

Conjecture 1. If G_k is prime then $|k|$ is prime.

Conjecture 2. The number of primes in $\{G_1\}$ is infinite.

The known primes are $G_{-5} = 79$, $G_{-1} = 2$, $G_1 = 3$, $G_7 = 263$. G_{31} may be prime.

The sequence of G_{-k} is interesting. The first thirteen G_{-k} numbers are placed immediately below their corresponding G_k numbers beginning with $k = 1$ in both cases.

(21)
$$\begin{array}{l} 3, 6, 10, 21, 46, 108, 263, 658, 1674, 4305, 11146, 28980, 75547, \cdots \\ 2, 9, 10, 42, 79, 252, 582, 1645, 4106, 11070, 28459, 75348, 195898, \cdots . \end{array}$$

A glance at these G numbers provide another symmetry property,

(22) $$G_{-2n} - G_{2n} = F_{4n} \quad \text{and} \quad G_d + G_{-d} = L_{2d} + 2 \quad \text{for } d \text{ odd.}$$

And more generally it is rather easy to show via (20) that

(23) $$G_{j,-2n} - G_{j,2n} = P^*_{j,2n}(P_{j,2n+1} - P_{j,2n-1}) = jP_{j,4n}$$

(24) $$G_{j,d} + G_{j,-d} = P^*_{j,2d} + 2 \quad \text{for } d \text{ odd}$$

DIFFERENCES OF G_k

We need the following:

(25) $$\nabla^k H_n = H_{n-2k} \quad \text{and so} \quad \nabla^k H_k = H_{-k}$$

(26) $$\nabla^{2k} B_n = B_{n-k} \quad \text{and} \quad \nabla^{2k+1} B_n = \nabla B_{n-k}$$

(27) $$\nabla^k A_n = \text{signum}(A_n)|A_{n+k}|,$$

where B_n is any bisection of H_n, and where (25) and (26) are easily derivable from

(28) $$H_{n+1} = H_n + H_{n-1}, \quad \text{any } H_0 \text{ and } H_1,$$

and where A_n is a two-point sequence with alternating signs satisfying

(29) $$A_{n+1} = -A_n + A_{n-1}$$

corresponding to $j = -1$ in (3), and signum is the sign function.

Then application of (25) and (26) to (1) immediately gives

(30) $$\nabla^k G_k = F_{k-1} + (-1)^k L_k,$$

which becomes $-F_{k+1}$ in the odd k case. Note that these numbers lie along a falling diagonal from $G_0 = 4$ in Table 2. Equation (30) introduces a significant simplicity into the G_k numbers. Note that (30) is reminiscent of the definition of the Bell numbers, to wit:

(31) $$\nabla^{n-1}\text{Bell}_n = \text{Bell}_{n-1}, \quad n \geq 2.$$

Likewise one may also show that

(32) $$\nabla^{k-1} G_k = F_{k-4} \quad \text{for odd } k \geq 3$$

and these numbers 1, 1, 2, 5, ⋯ are a bisection of the falling diagonal from $G_1 = 3$. Note that *all* falling diagonals are two bisected sequences, B_n, and satisfy for all k and all $n \geq 1$,

(33) $$\Delta^{n+4} G_k = 3\Delta^{n+2} G_k - \Delta^n G_k .$$

I did not expect to find upon glancing at the central differences of G_0 that they would be: −3, 19, −75, ⋯ almost Lucas numbers. We may write

(34) $$\delta^{2n} G_0 = \nabla^{2n} G_n = 1 + (-1)^n L_{3n} .$$

This may be easily derived from (1) with $j = 1$ by applying (25). The critical step is

$$\nabla^{2k} L_k = L_{k-4k} = L_{-3k}$$

according to (25). We obtain

(35a) $$\nabla^{2k-1} G_k = L_{-3k+2}, \quad k \geq 1$$

(35b) $$\nabla^{2k} G_k = L_{-3k} + F_{-1}, \quad k \geq 1$$

(35c) $$\nabla^{2k+1} G_k = L_{-3k-2} + F_{-2}, \quad k \geq 0,$$

where, of course, $F_{-2} = -1$ and $F_{-1} = 1$. Equations (35) prove what is obvious by looking at Table 2, namely if we make a zig-zag below the 4 entry we obtain the sequence: −1, 2, −3, 7, −12, 19, −29, 46, −75, 123, ⋯ which is almost the Lucas sequence. This makes the whole sequence easy to generate by hand. Finally the choice of letter for these sequences was Gould's [1] who suggested my name for them after seeing my paper [6].

The author appreciates some comments by Zeitlin [8] concerning (14) and (23). Zeitlin [7] has also pointed out that the subscript of the subscript of the last term of Eq. (12) of [6] should be $(k - 1)$ and not $(k - 2)$. This misprint is obvious from the expansion in (13) of [6].

Having found that the messy looking $G_{j,k}$ sequence actually satisfies the near Fibonacci relationships (10) and (12) and further that the Lucas numbers have made their presence known, I am impelled to write down an old haiku of mine in which even the numbers of syllables in each line, namely 3, 2, 5, 7 are themselves a Fibonacci sequence.

PHI

Multiply
Or add
We always reach phi
Symmetries we perpetrate.

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where the i^{th} column of C_n is the i^{th} row of Pascal's triangle adjusted to the main diagonal and the other entries are 0's. Find $C_n \cdot A_n^T$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

A. Let $B_n = A_n \cdot A_n^T$. Let a_{ij} and b_{ij} be the entries in the i^{th} row and j^{th} column of A_n and B_n, respectively. Similarly, let a_{ij}^T be the j^{th} entry of A_n^T. Then

$$a_{ij} = \binom{i-1}{j-1} \quad \text{if } i \geq j;$$
$$= 0 \quad \text{elsewhere;}$$

therefore,

$$a_{ij}^T = \binom{j-1}{i-1} \quad \text{if } i \leq j$$
$$= 0 \quad \text{elsewhere.}$$

[continued on page 183.]