PELLIAN DIOPHANTINE SEQUENCES

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1. INTRODUCTION

The so-called Pellian Diophantine equation is

$$x_{22}^2 - 2x_{12}^2 = 1$$

which can be generalized to

$$|x_{22}^2 - mx_{12}| = 1,$$

or

$$\text{abs.}\begin{vmatrix} x_{22} & mx_{12} \\ x_{12} & x_{22} \end{vmatrix} = 1.$$

A generalization of this is in turn provided by

$$\text{(1.1)} \qquad \text{abs.}\begin{vmatrix} x_{rr} & mx_{1r} & mx_{2r} & \cdots & mx_{r,r-1} \\ x_{r-1,r} & x_{rr} & mx_{1r} & \cdots & mx_{r,r-2} \\ & & \cdots & & \\ x_{1r} & x_{2r} & x_{3r} & \cdots & x_{rr} \end{vmatrix} = 1.$$

The aim of this paper is to construct a solution for this generalized Pellian Diophantine equation. The approach adopted is less general than that of Bernstein [1] but is, in a sense, more direct. For encouragement with an earlier draft of this paper thanks are due to Bernstein, whose works on pyramidal Diophantine equations [3] and the Jacobi-Perron algorithm [2] should be seen for further extensions. We designate the determinant in Eq. (1.1) by

$$D(m; x_{1r}, \cdots, x_{rr}).$$

2. SEQUENCES

We define sequences $\{W_{s,n}^{(r)}\}$ which satisfy the arbitrary order linear homogeneous recurrence relation

$$\text{(2.1)} \qquad W_{s,n}^{(r)} = \sum_{j=1}^{r} \binom{r}{j} D^{r-j} W_{s,n-j}^{(r)}, \qquad n > r,$$

where

$$D = [w], \quad w \text{ an } r^{th}\text{-degree irrational:}$$

$$w^r = m$$

$$= D^r + d, \quad m, D, d \in Z_+,$$

with boundary conditions determined by

$$W_{s,n}^{(r)} = \delta_{s,n+1} \quad \begin{cases} s \leqslant n+1 \\ 1 \leqslant n < r \end{cases}$$

$$W_{s,r}^{(r)} = D^{s-1}$$

$$\text{(2.2)} \qquad W_{s,r}^{(r)} = DW_{s-1,n}^{(r)} + W_{s-1,n-1}^{(r)}.$$

The initial values $W_{s,1}^{(r)}$, $s > 2$, have not been specified because they are not used in this development. They are readily determined from Eqs. (2.1) and (2.2) if required.

The table provides some examples of $W^{(2)}_{s,n}$ and $W^{(3)}_{s,n}$.

Each of the sequences can be expressed in terms of the fundamental sequence [6], $\{W^{(r)}_{1,n}\}$:

$$W^{(r)}_{s,n} = \sum_{j=0}^{s-1} \binom{s-1}{j} D^{s-j-1} W^{(r)}_{1,n-j}\,.$$

Proof. When $s = 1,2$, we have respectively

$$W^{(r)}_{1,n} = W^{(r)}_{1,n} \quad\text{and}\quad W^{(r)}_{2,n} = DW^{(r)}_{1,n} + W^{(r)}_{1,n-1}\,.$$

Suppose the result is true for $s = 1, 2, \cdots, t$.

$$W^{(r)}_{t+1,n} = DW^{(r)}_{t,n} + W^{(r)}_{t,n-1} = \sum_{j=0}^{t-1} \binom{t-1}{j} \{D^{t-j} W^{(r)}_{1,n-j} + D^{t-j-1} W^{(r)}_{1,n-j-1}\}$$

$$= \sum_{j=0}^{t} \left\{ \binom{t-1}{j} + \binom{t-1}{j-1} \right\} D^{t-j} W^{(r)}_{1,n-j} = \sum_{j=0}^{t} \binom{t}{j} D^{t-j} W^{(r)}_{1,n-j}\,,$$

as required

3. LEMMAS

We define matrices M, N_n:

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & rD & \binom{r}{2} D^2 & \cdots & rD^{r-1} \end{bmatrix},$$

$$N_n = [W^{(r)}_{\kappa,n+\rho}] \qquad 1 \leqslant \kappa, \quad \rho \leqslant r.$$

Lemma 1.

$$N_{n+1} = M^n N_1\,.$$

Proof. The result clearly follows from induction on n, since when $n = 1$,

$$MN_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \cdots & & \\ 0 & 0 & \cdots & 1 \\ 1 & rD & \cdots & rD^{r-1} \end{bmatrix} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \cdots & & \\ 0 & 0 & \cdots & 1 \\ 1 & W^{(r)}_{2,r+1} & \cdots & W^{(r)}_{r,r+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & W^{(r)}_{r,3} \\ & \cdots & & \\ 1 & W^{(r)}_{2,r+1} & \cdots & W^{(r)}_{r,r+1} \\ W^{(r)}_{1,r+2} & W^{(r)}_{2,r+2} & \cdots & W^{(r)}_{r,r+2} \end{bmatrix} = N_2\,.$$

$$N_3 = MN_2$$
$$= M^2 N_1, \text{ and so on.}$$

Lemma 2.

$$\det N_n = (-1)^{n(r-1)}.$$

Proof.

$$\det M = (-1)^{r-1} = \det N_1\,.$$

$$\det N_n = (-1)^{(r-1)(n-1)}(-1)^{r-1} = (-1)^{n(r-1)}.$$

Lemma 3.

$$\sum_{k=1}^{r} \sum_{j=0}^{r-k} \binom{r-k}{j} W^k D^j W^{(r)}_{i,n+j+k} = \sum_{k=1}^{r} \sum_{j=0}^{r-k} \binom{r-k}{j} W^{k-1} D^j W^{(r)}_{i+1,n+j+k}\,.$$

Proof. We consider coefficients of w:

$$\sum_{j=0}^{r-k-1}\binom{r-k-1}{j}D^jW^{(r)}_{i+1,n+j+k+1} = \sum_{j=0}^{r-k-1}\binom{r-k-1}{j}D^j(DW^{(r)}_{i,n+j+k+1}+W^{(r)}_{i,n+j+k})$$

$$= \sum_{j=0}^{r-k-1}\left\{\binom{r-k-1}{j}D^{j+1}W^{(r)}_{i,n+j+k+1}+\binom{r-k-1}{j}D^jW^{(r)}_{i,n+j+k}\right\}$$

$$= \sum_{j=0}^{r-k}\left\{\binom{r-k-1}{j-1}+\binom{r-k-1}{j}\right\}D^jW^{(r)}_{i,n+j+k}$$

$$= \sum_{j=0}^{r-k}\binom{r-k}{j}D^jW^{(r)}_{i,n+j+k}, \quad \text{as required .}$$

4. RESULT

Theorem. For $i, k = 1, 2, \cdots, r$,

$$x_{ik} = \sum_{j=0}^{r-k}\binom{r-k}{j}D^jW^{(r)}_{i,n+j+k}$$

are solutions of the Pellian Diophantine equation

$$1 = D(m; x_{1r}, \cdots, x_{rr}).$$

Proof. Lemma 3 becomes

$$\sum_{k=1}^{r} w^k x_{ik} = \sum_{k=1}^{r} w^{k-1}x_{i+1,k} \,. \tag{4.1}$$

$$(-1)^{n(r-1)} = \det N^n = \begin{vmatrix} W^{(r)}_{1,n+1} & W^{(r)}_{2,n+1} & \cdots & W^{(r)}_{r,n+1} \\ W^{(r)}_{1,n+2} & W^{(r)}_{2,n+2} & \cdots & W^{(r)}_{r,n+2} \\ & \cdots & & \\ W^{(r)}_{1,n+r} & W^{(r)}_{2,n+r} & \cdots & W^{(r)}_{r,n+r} \end{vmatrix}$$

$$= \begin{vmatrix} W^{(r)}_{1,n+1} + \sum_{j=1}^{r-1}\binom{r-1}{j}D^jW^{(r)}_{1,n+j+1} & \cdots & W^{(r)}_{r,n+1} + \sum_{j=1}^{r}\binom{r-1}{j}D^jW^{(r)}_{r,n+j+1} \\ W^{(r)}_{1,n+2} + \sum_{j=1}^{r-2}\binom{r-2}{j}D^jW^{(r)}_{1,n+j+2} & \cdots & W^{(r)}_{r,n+2} + \sum_{j=1}^{r}\binom{r-2}{j}D^jW^{(r)}_{r,n+k+2} \\ & \cdots & \\ W^{(r)}_{1,n+r-1} + DW^{(r)}_{1,n+r} & \cdots & W^{(r)}_{r,n+r-1} + DW^{(r)}_{r,n+r} \\ W^{(r)}_{1,n+r} & \cdots & W^{(r)}_{r,n+r} \end{vmatrix}$$

$$= \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{r1} \\ x_{12} & x_{22} & \cdots & x_{r2} \\ & \cdots & & \\ x_{1r} & x_{2r} & \cdots & x_{rr} \end{vmatrix} = D(m; x_{1r}, \cdots, x_{rr})$$

by equating coefficients of w^k in Eq. (4.1).

5. CONCLUSION

Consider, as examples: When $r = 2$, $m = 2$, we have

$$D = [\sqrt{2}] = 1, \quad \text{and} \quad x_{22} = W^{(2)}_{2,n+2}, \quad x_{12} = W^{(2)}_{1,n+2}.$$

When $n = 1$,

$$x_{22} = W^{(2)}_{2,3} = 3, \quad x_{12} = W^{(2)}_{1,3} = 2,$$

which satisfy

$$x_{22}^2 - mx_{12}^2 = 1;$$

when $n = 0$,

$$x_{22} = W^{(2)}_{2,2} = 1, \quad x_{12} = W^{(2)}_{1,2} = 1,$$

which satisfy

$$x_{22}^2 - mx_{12}^2 = -1.$$

The relevant recurrence relation is

$$W^{(2)}_{s,n} = 2DW^{(2)}_{s,n-1} + W^{(2)}_{s,n-2}.$$

When $r = 3$, $m = 9$, we have

$$D = [\sqrt[3]{9}] = 2, \quad \text{and} \quad x_{33} = W^{(3)}_{3,n+3}, \quad x_{23} = W^{(3)}_{2,n+3}, \quad x_{13} = W^{(3)}_{1,n+3}.$$

When $n = 0$,

$$x_{33} = W^{(3)}_{3,3} = 4, \quad x_{23} = W^{(3)}_{2,3} = 2, \quad x_{13} = W^{(3)}_{1,3} = 1,$$

which satisfy

$$x_{33}^3 + mx_{23}^3 + m^2x_{13}^3 - 3mx_{13}x_{23}x_{33} = 1.$$

The relevant recurrence relation is

$$W^{(3)}_{s,n} = 3D^2W^{(3)}_{s,n-1} + 3DW^{(3)}_{s,n-2} + W^{(3)}_{s,n-3}, \qquad n > 3.$$

There is scope for further research in generalizing the properties of the second-order Pellian sequence discussed by Horadam [5]. The use of the Jacobi-Perron Algorithm in this context should be studied first [2]. The other way of generalizing the Pellian equation, namely,

$$x^r - my^r = 1,$$

is still an open and challenging question as Bernstein [4] remarked.

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