SOME RESULTS FOR GENERALIZED BERNOULLI, EULER, STIRLING NUMBERS

L. TOSCANO
Messina, Italy

SUMMARY

The present paper is a continuation of researches begun by the author in previous publications [3, 4, 5] on three classes of generalized Bernoulli, Euler, Stirling numbers. And here, of course, will be proved some additional interesting results.

1. GENERALIZED BERNOULLI, EULER NUMBERS AND POLYNOMIALS

The generalized Bernoulli, Euler numbers in question, and the related polynomials, are defined by the series

$$f(t;h,w) = \frac{ht}{(1+wt)^{h/w}-1} = \sum_{n=0}^{\infty} B_{n;h,w}\frac{t^n}{n!}, \tag{1.1}$$

$$\varphi(t;h,w) = \frac{2h(1+wt)^{1/w}}{(1+wt)^{2h/w}+1} = \sum_{n=0}^{\infty} E_{n;h,w}\frac{t^n}{n!}, \tag{1.2}$$

$$F(x,t;h,w) = \frac{ht(1+wt)^{x/w}}{(1+wt)^{h/w}-1} = \sum_{n=0}^{\infty} B_{n;h,w}(x)\frac{t^n}{n!}, \tag{1.3}$$

$$\phi(x,t;h,w) = \frac{2h(1+wt)^{x/w}}{(1+wt)^{h/w}+1} = \sum_{n=0}^{\infty} E_{n;h,w}(x)\frac{t^n}{n!}, \tag{1.4}$$

where h and w are real parameters.

These series, for a correct treatment, will be considered in the neighborhood of the origin.

The explicit expressions of $B_{n;h,w}(x)$ for $n = 0, 1, \cdots, 5$ are

$$B_{0;h,w}(x) = 1,$$

$$B_{1;h,w}(x) = \tfrac{1}{2}(2x-h+w),$$

$$B_{2;h,w}(x) = x(x-h)+\frac{1}{6}(h^2-w^2),$$

$$B_{3;h,w}(x) = \tfrac{1}{2}x(x-h)(2x-h-3w)-\tfrac{1}{4}w(h^2-w^2),$$

$$B_{4;h,w}(x) = x(x-h)(x-2w)(x-h-2w)-\frac{1}{30}(h^2-w^2)(h^2-19w^2)$$

$$B_{5;h,w}(x) = x(x-h)[x^3-\frac{3}{2}(h+5w)x^2+\frac{1}{6}(h^2+45hw+110w^2)x+\frac{1}{6}h^3-\frac{55}{6}hw^2 - 15w^3]+\frac{1}{4}w(h^2-w^2)(h^2-9w^2).$$

And these can be deduced by the recurrent relation [4]

$$\sum_{r=0}^{n-1} \binom{n}{r} (-w)^{-r} \left(1 - \frac{h}{w}\right)_{n-r-1} B_{r;h,w}(x) = n(-x/w)_{n-1}, \quad n > 0,$$

where $(a)_0 = 1,\ (a)_r = a(a+1)\cdots(a+r-1)$.

The explicit expressions of $E_{n;h,w}(x)$ for $n = 0, 1, \cdots, 5$ are

$$E_{0;h,w}(x) = h,$$
$$E_{1;h,w}(x) = \tfrac{1}{2}h(2x-h),$$
$$E_{2;h,w}(x) = hx^2 - h(h+w)x + \tfrac{1}{2}h^2w,$$
$$E_{3;h,w}(x) = hx^3 - \frac{3}{2}h(h+2w)x^2 + hw(3h+2w)x + \frac{1}{4}h^2(h^2-4w^2),$$
$$E_{4;h,w}(x) = hx^4 - 2h(h+3w)x^3 + hw(9h+11w)x^2 + h(h^3 - 11hw^2 - 6w^3)x - \frac{3}{2}h^2w(h^2-2w^2),$$
$$E_{5;h,w}(x) = hx^5 - \frac{5}{2}h(h+4w)x^4 + 5hw(4h+7w)x^3 + \frac{5}{2}h(h^3 - 21hw^2 - 20w^3)x^2 - 2hw(5h^3 - 25hw^2 - 12w^3)x - \frac{1}{2}h^2\left(h^4 - \frac{35}{2}h^2w^2 + 24w^4\right).$$

And these can be deduced by the recurrent relation [4]

$$2E_{n;h,w}(x) + \sum_{r=0}^{n-1} \binom{n}{r} (-w)^{n-r} (-x/w)_{n-r} E_{r;h,w}(x) = 2h(-w)^n(-x/w)_n,$$

$n > 0$. For relations with generalized Bernoulli, Euler, polynomials, it is easy to see that generalized Bernoulli, Euler, numbers can be derived by the formulas

$$B_{n;h,w} = B_{n;h,w}(0), \qquad E_{n;h,w} = 2^n E_{n;h,w/2}(\tfrac{1}{2}).$$

The first six values of $E_{n;h,w}$ are given by

$$E_{0;h,w} = h, \qquad E_{1,h,w} = h(1-h),$$
$$E_{2;h,w} = h(1-2h) + h(h-1)w,$$
$$E_{3;h,w} = h(h-1)(2h^2+2h-1) + 3h(2h-1)w + 2h(1-h)w^2,$$
$$E_{4;h,w} = h(2h-1)(4h^2+2h-1) + 6h(1-h)(2h^2+2h-1)w + 11h(1-2h)w^2 + 6h(h-1)w^3,$$
$$E_{5;h,w} = 4h(1-h)(4h^4+4h^3-h^2-h-1) - 10(8h^4-4h^2+1)w + 35h(h-1)(2h^2+2h-1)w^2 + 50h(2h-1)w^3 + 24h(1-h)w^4.$$

Moreover, it will be useful to estimate also the expression

$$F_{n;h,w} = 2^n E_{n;h,w/2}(h/2).$$

And here, of course, we introduce the particular expressions for $n = 0, 1, \cdots, 5$:

$$F_{0;h,w} = h, \qquad F_{1;h,w} = 0, \qquad F_{2;h,w} = -h^3,$$
$$F_{3;h,w} = 3h^3w, \qquad F_{4;h,w} = h^3(5h^2-11w^2), \qquad F_{5;h,w} = -50h^3w(h^2-w^2).$$

The theory of generalized Bernoulli, Euler, numbers and polynomials was first investigated by R. Lagrange [1], L. Tanzi Cattabianchi [2], and later extensively in the author's paper [4].

If $h = 1,\ w = 0$, the numbers $B_{n;h,w}$, $E_{n;h,w}$, and the polynomials $B_{n;h,w}(x)$, $E_{n;h,w}(x)$, reduce to the ordinary Bernoulli, Euler, numbers and polynomials, generally defined by the generating expansions

$$\text{(1.5)} \qquad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi,$$

(1.6)
$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \qquad |t| < \pi/2,$$

(1.7)
$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi,$$

(1.8)
$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad |t| < \pi.$$

2. GENERALIZED STIRLING NUMBERS

The ordinary Stirling numbers of the first and second kind $s_{n,r}$, $S_{n,r}$, are defined by the initial values

$$s_{n,1} = (-1)^{n-1}(n-1)!, \qquad s_{n,n} = 1,$$
$$S_{n,1} = S_{n,n} = 1,$$

and the recurrences

$$s_{n,r} = s_{n-1,r-1} - (n-1)s_{n-1,r}, \qquad 1 < r < n,$$
$$S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}, \qquad 1 < r < n,$$

with

$$s_{n,0} = 0, \qquad S_{n,0} = 0,$$
$$s_{n,r} = 0, \qquad S_{n,r} = 0, \text{ provided } r > n.$$

In our paper [3], they have been generalized with the coefficients $a_{n,r}^{(u)}$ satisfying the recurrence

$$a_{n,r}^{(u)} = a_{n-1,r-1}^{(u)} - [n + r(u-1) - 1]a_{n-1,r}^{(u)}, \qquad 1 < r < n,$$

with

$$a_{n,1}^{(u)} = (-1)^{n-1}(u)_{n-1}, \qquad a_{n,n}^{(u)} = 1,$$
$$a_{n,0}^{(u)} = 0, \qquad a_{n,r}^{(u)} = 0 \text{ provided } r > n.$$

And the particular expressions of $a_{n,r}^{(u)}$ for $n = 1, 2, \cdots, 5$, $r = 1, 2, \cdots, 5$, are

$$a_{1,1}^{(u)} = 1, \qquad a_{2,1}^{(u)} = -u, \qquad a_{2,2}^{(u)} = 1,$$

$$a_{3,1}^{(u)} = (u)_2, \qquad a_{3,2}^{(u)} = -3u, \qquad a_{3,3}^{(u)} = 1,$$

$$a_{4,1}^{(u)} = -(u)_3, \qquad a_{4,2}^{(u)} = u(7u+4), \qquad a_{4,3}^{(u)} = -6u, \qquad a_{4,4}^{(u)} = 1,$$

$$a_{5,1}^{(u)} = (u)_4, \qquad a_{5,2}^{(u)} = -5u(u+1)(3u+2), \qquad a_{5,3}^{(u)} = 5u(5u+2),$$

$$a_{5,4}^{(u)} = -10u, \qquad a_{5,5}^{(u)} = 1.$$

Our paper [3] presents an extensive treatment of the coefficients $a_{n,r}^{(u)}$, and it is interesting to note here that

(2.1)
$$a_{n,r}^{(u)} = \frac{(-1)^{n-r}}{(u-1)^r r!} \underset{u-1}{\overset{r}{\Delta}} x(x)_n \quad \text{provided } x = 0,$$

where $\underset{v}{\Delta} x$ is the *descending* difference defined by the relation $\underset{v}{\Delta} x f(x) = f(x+v) - f(x)$,

(2.2)
$$a_{n,r}^{(u)} = \frac{(-1)^n}{(u-1)^r r!} \sum_{k=1}^{r} (-1)^k \binom{r}{k} (ku-k)_n ,$$

(2.3) $$(-1)^n(x)_n = \sum_{r=1}^{n} a_{n,r}^{(u)}(u-1)^r\left(\frac{x}{1-u}\right)_r ,$$

(2.4) $$(-1)^n(x)_n = \sum_{r=0}^{n} a_{n+1,r+1}^{(u)}(u-1)^r\left(\frac{x-u}{1-u}\right)_r ,$$

(2.5) $$a_{n,r}^{(u)} = \sum_{k=r}^{n} s_{n,k}S_{k,r}(1-u)^{k-r} ,$$

from which

(2.6) $$a_{n,r}^{(1)} = s_{n,r} ,$$

(2.7) $$\lim_{u\to\infty} [(1-u)^{r-n}a_{n,r}^{(u)}] = S_{n,r} ;$$

(2.8) $$a_{n,r}^{(0)} = \begin{cases} 0, r < n, \\ 1, r = n , \end{cases}$$

(2.9) $$a_{n,r}^{(2)} = (-1)^{n-r}\frac{n!}{r!}\binom{n-1}{r-1} ,$$

(2.10) $$a_{n,r}^{(-1)} = \frac{1}{2^{n-r}}\cdot\frac{n!}{r!}\binom{r}{n-r} , \qquad r \geqslant n/2,$$

(2.11) $$a_{n,r}^{(1/2)} = \frac{(-1)^{n-r}}{2^{2n-2r}}\cdot\frac{(2n-r-1)!}{(n-1)!}\binom{n-1}{r-1} .$$

For references and applications of the coefficients $a_{n,r}^{(u)}$ to the operators satisfying the condition of *permutableness of the second order,* see the more recent our paper [5].

3. PARTICULAR EXPANSIONS. n^{th} DERIVATIVE OF

$$y(t) = \frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i}, \quad \text{where} \quad i^2 = -1.$$

From (1.5), placing to the left member the term 1, $-t/2$ under the summation sign of the right member, we find, as it is well known, the expansion

(3.1) $$\tan t = \sum_{n=0}^{\infty} (-1)^n 2^{2n+2}(2^{2n+2}-1)B_{2n+2}\frac{t^{2n+1}}{(2n+2)!}, \qquad |t| < \pi/2 .$$

Now, an expansion analogous to (3.1) will be derived from (1.1), proceeding similarly. First of all we have

$$\frac{ht}{(1+wt)^{h/w}-1} - 1 + \tfrac{1}{2}(h-w)t = \frac{1}{2[(1+wt)^{h/w}-1]}(2ht + [(h-w)t-2][(1+wt)^{h/w}-1])$$
$$= \sum_{r=2}^{\infty} B_{r;h,w}\frac{t^r}{r!} .$$

At once, changing at first t with $4t$, w with $w/4$, and after t with $2t$, w with $w/2$, we obtain the expansions

$$\frac{1}{2[(1+wt)^{4h/w}-1]}(8ht + [4ht - wt - 2][(1+wt)^{4h/w}-1])$$
$$= \sum_{r=2}^{\infty} 2^{2r}B_{r;h,w/4}\frac{t^r}{r!} ,$$

$$\frac{1}{2[(1+wt)^{2h/w}-1]}(4ht+[2ht-wt-2][(1+wt)^{2h/w}-1]) = \sum_{r=2}^{\infty} 2^r B_{r;h,w/2}\frac{t^r}{r!},$$

from which it follows that

$$\sum_{r=2}^{\infty}(2^{2r}B_{r;h,w/4}-2^rB_{r;h,w/2})\frac{t^r}{r!} = \frac{1}{2[(1+wt)^{4h/w}-1]}(8ht+[4ht-wt-2][(1+wt)^{4h/w}-1]$$
$$-4ht[(1+wt)^{2h/w}+1]-[2ht-wt-2][(1+wt)^{4h/w}-1])$$
$$= \frac{ht[(1+wt)^{2h/w}-1]^2}{(1+wt)^{4h/w}-1} = \frac{ht[(1+wt)^{2h/w}-1]}{(1+wt)^{2h/w}+1}.$$

(3.2) $$\frac{h[(1+wt)^{2h/w}-1]}{(1+wt)^{2h/w}+1} = \sum_{n=0}^{\infty} 2^{n+2}(2^{n+2}B_{n+2;h,w/4}-B_{n+2;h,w/2})\frac{t^{n+1}}{(n+2)!}.$$

Moreover by (1.4), replacing x by $h/2$, w by $w/2$, t by $2t$, we obtain the other expansion

(3.3) $$\frac{2h(1+wt)^{h/w}}{(1+wt)^{2h/w}+1} = \sum_{n=0}^{\infty} 2^n E_{n;h,w/2}(h/2)\frac{t^n}{n!},$$

and since

$$\frac{h}{i}\cdot\frac{(1+wt)^{2h/w}-1}{(1+wt)^{2h/w}+1}+\frac{2h(1+wt)^{h/w}}{(1+wt)^{2h/w}+1} = \frac{h[(1+wt)^{h/w}+i]^2}{i[(1+wt)^{2h/w}+1]}$$
$$= \frac{h}{i}\cdot\frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i},$$

we deduce thus the interesting expansion

(3.4) $$\frac{h}{i}\cdot\frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i} = \sum_{n=0}^{\infty}\frac{2^{n+2}}{i}(2^{n+2}B_{n+2;h,w/4}-B_{n+2;h,w/2})\frac{t^{n+1}}{(n+2)!}+\sum_{n=0}^{\infty}2^nE_{n;h,w/2}(h/2)\frac{t^n}{n!}.$$

After this expansion and for the following, it will be to estimate the n^{th} derivative of

$$y(t) = \frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i}.$$

Now we consider two continuous and derivable functions $y=f(u)$, $u=\varphi(t)$, and the formula for derivatives of a composite function

$$\frac{d^ny}{dt^n} = \sum_{r=1}^{n}\frac{(-1)^r}{r!}\cdot\frac{d^ry}{du^r}\sum_{k=1}^{r}(-1)^k\binom{r}{k}u^{r-k}\frac{d^nu^k}{dt^n}, \qquad n>0.$$

With the assumption

$$y = f(u) = \frac{2i}{u-i}+1, \qquad u = \varphi(t) = (1+wt)^{h/w},$$

we arrive at

$$\frac{d^ny}{dt^n} = \sum_{r=1}^{n}\left[\frac{(-1)^r}{r!}\cdot\frac{2i(-1)^rr!}{(u-i)^{r+1}}\cdot\sum_{k=1}^{r}(-1)^k\binom{r}{k}u^{r-k}(-w)^n\left(\frac{-kh}{w}\right)_n u^{k-nw/h}\right]$$
$$= \frac{2i(-w)^n}{(1+wt)^n}\sum_{r=1}^{n}\left[\frac{(1+wt)^{rh/w}}{[(1+wt)^{h/w}-i]^{r+1}}\cdot\sum_{k=1}^{r}(-1)^k\binom{r}{k}\left(\frac{-kh}{w}\right)_n\right].$$

Whence, by (2.2), we deduce, for $n > 0$,

(3.5) $$\frac{d^n}{dt^n} \cdot \frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-1}$$

$$= \frac{2iw^n}{(1+wt)^n} \sum_{r=1}^{n} \frac{(-1)^r r! h^r}{w^r} \cdot a_{n,r}^{(1-h/w)} \cdot \frac{(1+wt)^{rh/w}}{[(1+wt)^{h/w}-i]^{r+1}} .$$

Successively, putting $t = 0$, we have

$$\left[\frac{d^n}{dt^n} \cdot \frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i}\right]_{t=0} = 2iw^n \sum_{r=1}^{n} \frac{(-1)^r r! h^r}{w^r} \cdot a_{n,r}^{(1-h/w)} \cdot \frac{1}{(1-i)^{r+1}} .$$

Moreover it is

$$\frac{1}{1-i} = \frac{1+i}{2} = \frac{1}{\sqrt{2}} (\cos \pi/4 + i \sin \pi/4),$$

$$\frac{1}{(1-i)^{r+1}} = \frac{1}{2^{(r+1)/2}} [\cos (r+1)\pi/4 + i \sin (r+1)\pi/4]$$

$$= \frac{1}{2^{(r+1)/2}} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4] ,$$

and, however, it follows that $(n > 0)$

(3.6) $$\left[\frac{d^n}{dt^n} \cdot \frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i}\right]_{t=0}$$

$$= w^n \sum_{r=1}^{n} \frac{(-1)^{r-1} r! h^r}{2^{(r-1)/2} w^r} \cdot a_{n,r}^{(1-h/w)} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4] .$$

4. FORMULAS FOR THE GENERALIZED BERNOULLI, EULER, STIRLING NUMBERS

We now, by (3.6), deduce the expansion of the function $y(t)$ into a series of powers of t,

$$y(t) = \sum_{n=0}^{\infty} \left[\frac{d^n y(t)}{dt^n}\right]_{t=0} \frac{t^n}{n!}$$

$$= i + \sum_{n=1}^{\infty} \frac{(wt)^n}{n!} \sum_{r=1}^{n} \frac{(-1)^{r-1} r! h^r}{2^{(r-1)/2} w^r} a_{n,r}^{(1-h/w)} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4] .$$

Hence, comparing with (3.4), we obtain the expansion

$$\sum_{n=1}^{\infty} 2^{n+1}(2^{n+1} B_{n+1;h,w/4} - B_{n+1;h,w/2}) \frac{t^n}{(n+1)!}$$

$$+ iE_{0;h,w/2}(h/2) + i \cdot \sum_{n=1}^{\infty} 2^n E_{n;h,w/2}(h/2) \frac{t^n}{n!} = ih$$

$$+ \sum_{n=1}^{\infty} \frac{(wt)^n}{n!} \sum_{r=1}^{n} \frac{(-1)^{r-1} r! h^{r+1}}{2^{(r-1)/2} w^r} a_{n,r}^{(1-h/w)} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4] ,$$

from which, separating real and imaginary parts, and equating the coefficients of t^n on both sides, we establish the interesting formulas

(4.1)
$$2^{n+1}(2^{n+1}B_{n+1;h,w/4} - B_{n+1;h,w/2}) = (n+1)\sum_{r=1}^{n} \frac{(-1)^{r-1}r!h^{r+1}w^{n-r}}{2^{(r-1)/2}} a_{n,r}^{(1-h/w)} \cos(r-1)\pi/4,$$

(4.2)
$$2^nE_{n;h,w/2}(h/2) = \sum_{r=1}^{n} \frac{(-1)^{r-1}r!h^{r+1}w^{n-r}}{2^{(r-1)/2}} a_{n,r}^{(1-h/w)} \sin(r-1)\pi/r,$$

both for $n > 0$. They realize the principal objective of the present paper.

5. PARTICULAR FORMULAS

In this section we shall indicate some special cases of (4.1), (4.2).

(a) If $h = 1$, $w = 0$, the generalized numbers $B_{n;h,w}$, $E_{n;h,w}$, reduce to the ordinary Bernoulli, Euler, numbers, while

$$\lim_{w\to\infty} [w^{n-r}a_{n,r}^{(1-1/w)}] = S_{n,r}.$$

Moreover, it is $B_{2n+1} = 0$ for $n > 0$, $E_{2n+1} = 0$. Consequently, by (4.1), (4.2), we deduce the formulas

(5.1)
$$\frac{2^{2n+1}(2^{2n+2}-1)}{n+1} B_{2n+2} = \sum_{r=1}^{2n+1} \frac{(-1)^{r-1}r!}{2^{(r-1)/2}} S_{2n+1,r}\cos(r-1)\pi/4,$$

(5.2)
$$\sum_{r=1}^{2n+1} \frac{(-1)^{r-1}r!}{2^{(r-1)/2}} S_{2n+1,r}\sin(r-1)\pi/4 = 0,$$

(5.3)
$$\sum_{r=1}^{2n} \frac{(-1)^{r-1}r!}{2^{(r-1)/2}} S_{2n,r}\cos(r-1)\pi/4 = 0, \qquad n > 0,$$

(5.4)
$$E_{2n} = \sum_{r=2}^{2n} \frac{(-1)^{r-1}r!}{2^{(r-1)/2}} S_{2n,r}\sin(r-1)\pi/4, \qquad n > 0.$$

Equations (5.1) and (5.4) are two additional formulas concerning ordinary Bernoulli, Euler, Stirling numbers.

(b) If $w = h$, we have (2.8)

$$a_{n,r}^{(1-h/w)} = \begin{cases} 0, r < n, \\ 1, r = n, \end{cases}$$

therefore, (4.1) reduces to

(5.5)
$$2^{n+1}(2^{n+1}B_{n+1;h,h/4} - B_{n+1;h,h/2}) = \frac{(n+1)!(-h)^{n+1}}{2^{(n-1)/2}} \cos(n-1)\pi/4.$$

Moreover, by the recurrent relation for $B_{n;h,h/2}$, it follows that

$$B_{n;h,h/2} = -\frac{nh}{4} B_{n-1;h,h/2},$$

from which

(5.6) $$B_{n;h,h/2} = \frac{n!(-h)^n}{2^{2n}} .$$

Consequently, by (5.5) we deduce

(5.7) $$B_{n;h,h/4} = \frac{n!(-h)^n}{2^{3n}} [1 + 2^{(n+2)/2} \sin (n\pi/4)] .$$

This formula can be derived by the recurrent relation

$$B_{n;h,h/4} + \frac{3nh}{8} B_{n-1;h,h/4} + \frac{n(n-1)h^2}{16} B_{n-2;h,h/4} + \frac{n(n-1)(n-2)h^3}{256} B_{n-3;h,h/4} = 0, \qquad n > 0,$$

easily transformable in other forms to constant coefficients.

(c) If $w = -h$, we have (2.9)

$$a_{n,r}^{(1-h/w)} = (-1)^{n-r} \frac{n!}{r!} \binom{n-1}{r-1} ,$$

and (4.1) reduces to

(5.8) $$2^{n+1} (2^{n+1} B_{n+1;h,-h/4} - B_{n+1;h,-h/2}) = (n+1)!h^{n+1} \sum_{r=1}^{n} \frac{(-1)^{r-1}\binom{n-1}{r-1}}{2^{(r-1)/2}} \cos (r-1)\pi/4, \qquad n > 0.$$

Moreover, it is [4]

$$B_{n;h,-w} = (-1)^n B_{n;h,w} ,$$

and comparing (5.5) with (5.8) we have the identity, for $n > 0$,

(5.9) $$\sum_{r=1}^{n} (-1)^{r-1} \binom{n-1}{r-1} 2^{(n-r)/2} \cos (r-1)\pi/4 = \cos (n-1)\pi/4 .$$

Putting into (4.2) at first $w = h$ and after $w = -h$, and remembering that [4]

$$E_{n;h,-w}(h/2) = (-1)^n E_{n;h,w}(h/2) ,$$

we prove the identity

(5.10) $$\sum_{r=2}^{n} (-1)^r \binom{n-1}{r-1} 2^{(n-r)/2} \sin (r-1)\pi/4 = \sin (n-1)\pi/4 ,$$

for $n > 1$.

(d) If $w = h/2$, we have (2.10)

$$a_{n,r}^{(1-h/w)} = \frac{1}{2^{n-r}} \cdot \frac{n!}{r!} \binom{r}{n-r} , \qquad r \geq n/2 ,$$

and (4.1) becomes

(5.11) $$2^{n+1} (2^{n+1} B_{n+1;h,h/8} - B_{n+1;h,h/4}) = \frac{(n+1)!h^{n+1}}{2^{2n}} \sum_{r \geq n/2}^{n} (-1)^{r-1} \binom{r}{n-r} 2^{(3r+1)/2} \cos (r-1)\pi/4 .$$

Consequently, returning to (5.7), it follows

(5.12) $$2^{2n+2}B_{n+1;h,h/8} = \frac{(n+1)!h^{n+1}}{2^{2n+2}}\left((-1)^{n+1}[1+2^{(n+3)/2}\sin(n+1)\pi/4]\right.$$

$$\left. + \sum_{r\geqslant n/2}^{n}(-1)^{r-1}\binom{r}{n-r}2^{(3r+5)/2}\cos(r-1)\pi/4\right).$$

(e) If $w = 2h$, we have (2.11)

$$a_{n,r}^{(1-h/w)} = \frac{(-1)^{n-r}}{2^{2n-2r}}\cdot\frac{(2n-r-1)!}{(n-1)!}\cdot\binom{n-1}{r-1}.$$

Then (4.1), by (5.6) and the relation

$$B_{n;h,h} = 0 \quad \text{for} \quad n > 0,$$

reduces to the identity, for $n > 0$,

(5.13) $$\sum_{r=1}^{n} r!(2n-r-1)!\binom{n-1}{r-1}2^{(r+1)/2}\cos(r-1)\pi/4 = 2^n(n-1)!n!.$$

6. A DERIVATIVE FORMULA

Putting

$$P_{0;h,w} = i,$$

$$P_{n;h,w} = \frac{2^{n+1}}{(n+1)h}\cdot(2^{n+1}B_{n+1;h,w/4} - B_{n+1;h,w/2}) + \frac{i2^n}{h}E_{n;h,w/2}(h/2), \qquad n > 0,$$

the expansion (3.4) can be written in the form

(6.1) $$F(t) = \frac{(1+wt)^{h/w}+i}{(1+wt)^{h/w}-i} = \sum_{n=0}^{\infty} P_{n;h,w}\frac{t^n}{n!}.$$

Moreover, it is not difficult to show that the function $F(t)$ satisfies the functional equation

(6.2) $$\frac{2(1+wt)}{h}\cdot\frac{dF(t)}{dt} = 1 - F^2(t),$$

from which the recurrent relation follows,

(6.3) $$2P_{n+1;h,w} + 2nwP_{n;h,w} + h\sum_{r=0}^{n}\binom{n}{r}P_{r;h,w}P_{n-r;h,w} = 0, \qquad n > 0.$$

If $h = 1$, $w = 0$, we have the interesting connections with the ordinary Bernoulli, Euler numbers

(6.4) $$P_{2n;1,0} = iE_{2n}, \qquad P_{2n+1;1,0} = \frac{2^{2n+1}(2^{2n+2}-1)}{n+1}B_{2n+2},$$

and from (6.3) we obtain, in conclusion, the special formulas

(6.5) $$\sum_{r=0}^{n-1}\binom{2n}{2r+1}\frac{2^{2n}(2^{2r+2}-1)(2^{2n-2r}-1)}{(r+1)(n-r)}B_{2r+2}B_{2n-2r} - \sum_{r=0}^{n}\binom{2n}{2r}E_{2r}E_{2n-2r}$$

$$+ \frac{2^{2n+2}(2^{2n+2}-1)}{n+1}B_{2n+2} = 0, \qquad n > 0,$$

(6.6) $$E_{2n+2} = \sum_{r=0}^{n} \binom{2n+1}{2r+1} \frac{2^{2r+1}(1-2^{2r+2})}{r+1} B_{2n+2}E_{2n-2r}, \qquad n \geq 0.$$

REFERENCES

1. R. Lagrange, "Mémoire sur les Suites de Polynômes," *Acta Math.,* 51 (1928), pp. 201–309.
2. L. Tanzi Cattabianchi, "Numeri e polinomi di Bernoulli parametrizzati," *Riv. Mat. Univ. Parma,* (2)4(1963), pp. 281–291.
3. L. Toscano, "Numeri di Stirling generalizzati, operatori differenziali e polinomi ipergeometrici," *Commentat. Pontificia Ac. Sci.,* 3(1939), pp. 721–757.
4. L. Toscano, "Polinomi e numeri di Bernoulli e di Eulero parametrizzati," *Matematiche,* 22(1967), pp. 68–91.
5. L. Toscano, "Numeri di Stirling generalizzati e operatori permutabili di secondo ordine," *Mathematiche*, 24(1969) pp. 492–518.

****

EXPANSION

PAMILLA GRAVES SMITH
1609 Main Street, Evanston, Wyoming 82930

As every science, save one, is modified and cast aside,
While mathematics only is built upon and grows,
So, too, my life's whole whims and whimsies pied
See their demise, while my regard for you goes
On. Like the Sieve of Eratosthenes, you sift
My drifting days and sort the prime.

As determinants reflect a change, I mirror-image you
And, palindromic, backward-forward go, from autumn into spring.
Approaching the limit of joy, you bring a rate of change
Which grows in my heart proportionate to you. Your range
Is my domain. By you, my worthiness a proof shows,
As solid as geometry, as crystalline as snows,
As coming-now as spring.

I am subset of you.
Happily, with you no negative numbers can deride
My existence, that foolish enterprise of sensibility;
Instead, a proper fraction of civility
Is mine. By power of example, exponent of grace,
You multiply and lace my life with life. The race
Is mine! Cantor-like you lift
Me to infinity sublime
And grant me a number prime.