A RECURRENCE SUGGESTED BY A COMBINATORIAL PROBLEM

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SECTION 1

Recurrences of the following kind occur in connection with a certain combinatorial problem (see §5 below). Let $e_1, \ldots, e_n$ be non-negative integers and q a parameter. Consider the recurrence

$$F(e_1, \ldots, e_n) = \sum_{j=1}^{n} q^{jN} F(e_1 - \delta_{1j}, \ldots, e_n - \delta_{nj}), \tag{1.1}$$

where

$$N = e_1 + \cdots + e_n, \tag{1.2}$$

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j), \end{cases} \tag{1.3}$$

$$F(0, \ldots, o) = 1 \tag{1.4}$$

and $F(e_1, \ldots, e_n) = 0$ if any $e_i < 0$.

Note that, for $q = 1$, (1.1) reduces to

$$F(e_1, \ldots, e_n) = \sum_{j=1}^{n} F(e_1 - \delta_{1j}, \ldots, e_n - \delta_{nj})$$

and $F(e_1, \ldots, e_n)$ becomes the multinomial coefficient

$$\frac{(e_1 + e_2 + \cdots + e_n)!}{e_1!e_2! \ldots e_n!}.$$

If we put

$$\boldsymbol{e} = (e_1, \ldots, e_n), \quad \delta_j = (\delta_{1j}, \ldots, \delta_{nj}), \tag{1.5}$$

then (1.1) becomes

$$F(\boldsymbol{e}) = \sum_{j=1}^{n} q^{jN} F(\boldsymbol{e} - \delta_j). \tag{1.6}$$

For $n = 1$, the recurrence (1.1) is simply

$$F(N) = q^N F(N - 1), \quad F(0) = 1. \tag{1.7}$$

The solution of (1.7) is immediate, namely

$$F(N) = q^{\frac{1}{2}N(N+1)}. \tag{1.8}$$

For $n = 2$, the situation is less simple. We take

$$F(e_1, e_2) = q^N F(e_1 - 1, e_2) + q^{2N} F(e_1, e_2)(N = e_1 + e_2). \tag{1.9}$$

Iteration of (1.9) gives

$$\begin{aligned} F(e_1, e_2) &= q^{2N-1}F(e_1 - 2, e_2) + q^{3N-2}(1+q)F(e_1 - 1, e_2 - 1) + q^{4N-2}F(e_1, e_2 - 2) \\ &= q^{3N-3}F(e\ \ - 3, e_2) + q^{4N-5}(1+q+q^2)F(e_1 - \ \ , e_2 - 1) \\ &\qquad + q^{5N-6}(1+q+q^2)F(e_1 - 1, e_2 - 2) + q^{6N-6}F(e_1, e_2 - 3). \end{aligned}$$

It is helpful to isolate the exponents as indicated in the following table.

$m \backslash r$	0	1	2	3	4	5
0	1					
1	N	$2N$				
2	$2N - 1$	$3N - 2$	$4N - 2$			
3	$3N - 3$	$4N - 5$	$5N - 6$	$6N - 6$		
4	$4N - 6$	$5N - 9$	$6N - 11$	$7N - 12$	$8N - 12$	
5	$5N - 10$	$6N - 14$	$7N - 17$	$8N - 19$	$9N - 20$	$10N - 20$

The special results above suggest that generally, for $m \geq 0$,

$$F(e_1, e_2) = \sum_{r+s=m} \begin{bmatrix} m \\ r \end{bmatrix} q^{(m+r)N - m(m-1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r), \tag{1.10}$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q^{m-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)}. \tag{1.11}$$

It follows from (1.11) that

$$\begin{bmatrix} m \\ r \end{bmatrix} q^r + \begin{bmatrix} m \\ r-1 \end{bmatrix} = \begin{bmatrix} m+1 \\ r \end{bmatrix}. \tag{1.12}$$

For $m = 1$, (1.10) reduces to (1.9). Assume that (1.10) holds for all $m \leq M$. Then by (1.9)

$$\begin{aligned}
F(e_1, e_2) &= \sum_{r+s=M} \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r)N - M(M-1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r) \\
&= \sum_{r+s=M} \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r)N - M(M-1) + \frac{1}{2}s(s-1)} \Big\{ q^{N-r-s} F(e_1 - s - 1, e_2 - r) \\
&\qquad + q^{2(N-r-s)} F(e_1 - s, e_2 - r - 1) \Big\} \\
&= \sum_{r+s=M} \Big\{ \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r+1)N - M^2 + \frac{1}{2}s(s-1)} F(e_1 - s - 1, e_2 - r) \\
&\qquad + \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r+2)N - M(M+1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r - 1) \Big\} \\
&= \sum_{r+s=M+1} \Big\{ \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r+1)N - M^2 + \frac{1}{2}(s-1)(s-2)} \\
&\qquad + \begin{bmatrix} M \\ r-1 \end{bmatrix} q^{(M+r+1)N - M(M+1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r) \Big\} \\
&= \sum_{r+s=M+1} q^{(M+r+1)N - M(M+1) + \frac{1}{2}s(s-1)} \Big\{ \begin{bmatrix} M \\ r \end{bmatrix} q^{M-s+1} \\
&\qquad + \begin{bmatrix} M \\ r-1 \end{bmatrix} \Big\} F(e_1 - s, e_2 - r) \\
&= \sum_{r+s=M+1} \begin{bmatrix} M+1 \\ r \end{bmatrix} q^{(M+r+1)N - M(M+1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r),
\end{aligned}$$

by (1.12). Thus (1.10) holds for $M + 1$ and therefore for all $m \geq 0$.

For $m = N$, (1.10) reduces to

$$F(s, r) = q^{(2r+s)(r+s) - (r+s)(r+s-1) + \frac{1}{2}s(s-1)} \begin{bmatrix} r+s \\ r \end{bmatrix}.$$

Simplifying and interchanging r and s, we get

$$F(r, s) = q^{\frac{1}{2}r(r-1) + (r+s)(s+1)} \begin{bmatrix} r+s \\ r \end{bmatrix}. \tag{1.13}$$

By a familiar identity, (1.13) gives

$$\sum_{r=0}^{m} q^{-m(m-r+1)} F(r, m - r) x^r = (1 + x)(1 + qx) \cdots (1 + q^{m-1}x). \tag{1.14}$$

SECTION 2

The case $n = 3$ of (1.1) is more difficult. We have

$$\begin{aligned} F(e_1, e_2, e_3) = q^N F(e_1 - 1, e_2, e_3) + q^{2N} F(e_1, e_2 - 1, e_3) \\ + q^{3N} F(e_1, e_2, e_3 - 1). \end{aligned} \tag{2.1}$$

Iteration gives

$$\begin{aligned}F(e_1, e_2, e_3) &= q^N\Big(q^{N-1}F(e_1-2, e_2, e_3) + q^{2N-2}F(e_1-1, e_2-1, e_3)\\ &\quad + q^{3N-3}F(e_1-1, e_2, e_3-1)\Big) + q^{2N}\Big(q^{N-1}F(e_1-1, e_2-1, e_3)\\ &\quad + q^{2N-2}F(e_1, e_2-2, e_3) + q^{3N-3}F(e_1, e_2-1, e_3-1)\Big)\\ &\quad + q^{3N}\Big(q^{N-1}F(e_1-1, e_2, e_3-1) + q^{2N-2}F(e_1, e_2-1, e_3-1)\\ &\quad + q^{3N-3}F(e_1, e_2, e_3-2)\Big)\\ &= q^{2N-1}F(e_1-2, e_2, e_3) + q^{3N-2}(1+q)F(e_1-1, e_2-1, e_3)\\ &\quad + q^{4N-3}(1+q)F(e_1-1, e_2, e_3-1) + q^{4N-2}F(e_1, e_2-2, e_3)\\ &\quad + q^{5N-3}(1+q)F(e_1, e_2-1, e_3-1) + q^{6N-3}F(e_1, e_2, e_3-2).\end{aligned}$$

A second iteration gives

$$\begin{aligned}F(e_1, e_2, e_3) &= q^{3N-3}F(e_1-3, e_2, e_3) + q^{6N-6}F(e_1, e_2-3, e_3)\\ &\quad + q^{9N-9}F(e_1, e_2, e_3-3)\\ &\quad + q^{6N-8}(1+2q+2q^3+q^4)F(e_1-1, e_2-1, e_3-1)\\ &\quad + q^{4N-5}(1+q+q^2)F(e_1-2, e_2-1, e_3)\\ &\quad + q^{5N-7}(1+q^2+q^4)F(e_1-2, e_2, e_3-1)\\ &\quad + q^{5N-6}(1+q^2)F(e_1-1, e_2-2, e_3)\\ &\quad + q^{7N-9}(1+q^2+q^4)F(e_1-1, e_2, e_3-2)\\ &\quad + q^{7N-8}(1+q+q^2)F(e_1, e_2-2, e_3-1)\\ &\quad + q^{8N-9}(1+q+q^2)F(e_1, e_2-1, e_3-2).\end{aligned}$$

It follows from the above that

$$F(1,0,0) = q,\ F(0,1,0) = q^2,\ F(0,0,1) = q^3, \tag{2.2}$$

$$\begin{cases}F(2,0,0) = q^3,\ F(0,2,0) = q^6,\ F(0,0,2) = q^9\\ F(1,1,0) = q^4(1+q),\ F(1,0,1) = q^5(1+q^2),\ F(0,1,1) = q^7(1+q),\end{cases} \tag{2.3}$$

$$\begin{cases}F(3,0,0) = q^6,\ F(0,3,0) = q^{12},\ F(0,0,3) = q^{18}\\ F(2,1,0) = q^7(1+q+q^2),\ F(2,0,1) = q^8(1+q^2+q^4)\\ F(0,2,1) = q^{13}(1+q+q^2),\ F(1,2,0) = q^9(1+q+q^2)\\ F(1,0,2) = q^{12}(1+q^2+q^4),\ F(0,1,2) = q^{15}(1+q+q^2)\\ F(1,1,1) = q^{10}(1+2q+2q^3+q^4).\end{cases} \tag{2.4}$$

It is convenient to write (2.1) in operational form. Define the operators E_1^{-1}, E_2^{-1}, E_3^{-1} by means of

$$\begin{aligned}E_1^{-1}\phi(e_1, e_2, e_3) &= \phi(e_1-1, e_2, e_3),\ E_2^{-1}\phi(e_1, e_2, e_3)\\ &= \phi(e_1, e_2-1, e_3),\ E_3^{-1}\phi(e_1, e_2, e_3) = \phi(e_1, e_2, e_3-1)\end{aligned} \tag{2.5}$$

and put

$$\Omega = q^N E_1^{-1} + q^{2N} E_2^{-1} + q^{3N} E_3^{-1} \qquad (N = e_1 + e_2 + e_3). \tag{2.6}$$

Then (2.1) becomes

$$F(e_1, e_2, e_3) = \Omega F(e_1, e_2, e_3) \qquad (N > 0). \tag{2.7}$$

For $m \geq 0$ we may write

$$\Omega^m = \sum_{r+s+t=m} q^{(r+2s+3t)N} C(r, s, t) E_1^{-r} E_2^{-s} E_3^{-t}, \tag{2.8}$$

where $C(r, s, t)$ is independent of N. Moreover

$$C(0, 0, 0) = 1 \tag{2.9}$$

and $C(r, s, t) = 0$ if any one of $r, s, t = 0$.

By (2.7) and (2.8),

$$F(e_1, e_2, e_3) = \Omega^N F(e_1, e_2, e_3)$$
$$= \sum_{r+s+t=N} q^{(e_1+2e_2+3e_3)N} C(r, s, t) F(e_1 - r, e_2 - s, e_3 - t),$$

so that

$$F(e_1, e_2, e_3) = q^{(e_1+2e_2+3e_3)N} C(e_1, e_2, e_3). \tag{2.10}$$

Hence (2.8) becomes

$$\Omega^m = \sum_{r+s+t=m} q^{(r+2s+3t)(N-m)} F(r, s, t) E_1^{-r} E_2^{-s} E_3^{-t} \tag{2.11}$$
$$(N = e_1 + e_2 + e_3,\ 0 \leq m \leq N).$$

Since

$$F(e_1, e_2, e_3) = \Omega^m F(e_1, e_2, e_3) \qquad (0 \leq m \leq N),$$

it therefore follows from (2.11) that

$$F(e_1, e_2, e_3) = \sum_{r+s+t=m} q^{(r+2s+3t)(N-m)} F(r, s, t) F(e_1 - r, e_2 - s, e_3 - t). \tag{2.12}$$

This may be written in a more symmetrical form:

$$F(a, b, c) = \sum_{\substack{r+r'=a \\ s+s'=b \\ t+t'=c \\ r+s+t=m}} q^{(r+2s+3t)(r'+s'+t')} F(r, s, t) F(r', s', t') \tag{2.13}$$
$$(0 \leq m \leq a + b + c),$$

with a, b, c, m fixed.

For example, with $a = b = c = 1$, $m = 2$, (2.13) gives

$$\begin{aligned}F(1, 1, 1) &= q^5 F(0, 1, 1)F(1, 0, 0) + q^4 F(1, 0, 1)F(0, 1, 0) + q^3 F(1, 1, 0)F(0, 0, 1)\\ &= q^{13}(1+q) + q^{11}(1+q^2) + q^{10}(1+q)\\ &= q^{10}(1+2q+2q^3+q^4).\end{aligned}$$

Note that with $a = b = c = 1$, $m = 1$, we get

$$\begin{aligned}F(1, 1, 1) &= q^2 F(1, 0, 0)F(0, 1, 1) + q^4 F(0, 1, 0)F(1, 0, 1) + q^6 F(0, 0, 1)F(1, 1, 0)\\ &= q^{10}(1+q) + q^{11}(1+q^2) + q^{13}(1+q)\\ &= q^{10}(1+2q+2q^3+q^4).\end{aligned}$$

Indeed (2.13) is not completely symmetrical in appearance. If we put $m' = r' + s' + t'$, then (2.13) yields

$$F(a, b, c) = \sum_{\substack{r'+r=a\\ s'+s=b\\ t'+t=c\\ r'+s'+t'=m}} q^{(r'+2s'+3t')(r+s+t)} F(r', s', t')F(r, s, t). \tag{2.14}$$

The equivalence of (2.13) and (2.14) can be verified directly by merely interchanging the roles of the primed and unprimed letters in (2.13).

By means of (2.13) a number of special values are easily computed. For example we have

$$\begin{cases} F(a, 0, a) = q^{a-1}F(1, 0, 0)F(a-1, 0, 0)\\ F(0, b, 0) = q^{2(b-1)}F(0, 1, 0)F(0, b-1, 0)\\ F(0, 0, c) = q^{3(c-1)}F(0, 0, 1)F(0, 0, c-1).\end{cases}$$

It then follows that

$$f(a, 0, 0) = q^{\frac{1}{2}a(a+1)}, \quad f(0, b, 0) = q^{b(b+1)}, \quad f(0, 0, c) = q^{\frac{3}{2}c(c+1)}. \tag{2.15}$$

As another example

$$F(a, 1, 0) = q^a F(1, 0, 0)F(\ -1, 1, 0) + q^{2a} F(0, 1, 0)F(a, 0, 0)$$

and we find that

$$F(a, 1, 0) = q^{\frac{1}{2}(a^2+3a+4)}(1+q+\cdots+q^a). \tag{2.16}$$

Similarly

$$F(0, 1, a) = q^{2a}F(0, 1, 0)F(0, 0, a) + q^{3a}F(0, 0, 1)F(0, 1, a-1),$$

which gives

$$F(0, 1, a) = q^{\frac{1}{2}(a+1)(3a+4)}(1+q+\cdots+q^a). \tag{2.17}$$

Also

$$\begin{cases} F(a,\ 0,\ 1) = q^{\frac{1}{2}(a^2+3a+4)}(1+q^2+\cdots+q^{2a}) \\ F(1,\ 0,\ a) = q^{\frac{1}{2}(a+1)(3a+2)}(1+q^2+\cdots+q^{2a}) \end{cases} \tag{2.18}$$

$$\begin{cases} F(1,\ a,\ 0) = q^{(a+1)^2}(1+q+\cdots+q^a) \\ F(0,\ a,\ 1) = q^{a^2+3a+3}(1+q+\cdots+q^a). \end{cases} \tag{2.19}$$

Note that it follows from (2.16), (2.17), (2.18), and (2.19) that

$$\begin{cases} F(0,\ 1,\ a) = q^{a(a+2)}F(a,\ 1,\ 0) \\ F(1,\ 0,\ a) = q^{a(a-1)}F(a,\ 0,\ 1) \\ F(0,\ a,\ 1) = q^{a(a+2)}F(1,\ a,\ 0). \end{cases} \tag{2.20}$$

SECTION 3

It is evident from (2.1) that $F(a,\ b,\ c)$ is a polynomial in q with non-negative integral coefficients. Put

$$f(a,\ b,\ c) = \deg F(a,\ b,\ c), \tag{3.1}$$

the degree of $F(a,\ b,\ c)$. To evaluate $f(a,\ b,\ c)$ we use (2.1):

$$F(a,\ b,\ c) = q^N F(a-1,\ b,\ c) + q^{2N}F(a,\ b-1,\ c) + q^{3N}F(a,\ b,\ c-1) \quad (N = a+b+c).$$

Then

$$f(a,\ b,\ c) = \max\left\{N + f(a-1,\ b,\ c),\ 2N + f(a,\ b-1,\ c),\ 3N + f(a,\ b,\ c-1)\right\}. \tag{3.2}$$

In particular

$$f(a,\ b,\ c) \geq 3N + f(a,\ b,\ c-1) \qquad (c > 0),$$

so that

$$f(a,\ b,\ c) \geq 3N + \ (N-1) + \cdots + 3(N-c+1) + f(a,\ b,\ 0).$$

Since, by (3.2),

$$f(a,\ b,\ 0) \geq 2(a+b) + f(a,\ b-1,\ 0),$$

we get

$$f(a,\ b,\ c) \geq 3N + 3(N-1) + \cdots + 3(N-c+1) + 2(N-c) + 2(N-c-1) + \cdots + 2(N-c-b+1) + f(a,\ 0,\ 0).$$

Hence, by (2.15)

$$f(a, b, c) \geq \frac{1}{2}a(a+1) + b(b+1) + \frac{3}{2}c(c+1) + 2ab + 3ac + 3bc. \tag{3.3}$$

We shall prove that, in fact,

$$f(a, b, c) = \frac{1}{2}a(a+1) + b(b+1) + \frac{3}{2}c(c+1) + 2ab + 3ac + 3bc. \tag{3.4}$$

This is evidently true for $a+b+c = 0, 1, 2, 3$. Assume that (3.4) holds for $a+b+c < M$. By (3.4), with $a+b+c = M$, we have

$$f(a, b, c-1) + a + b + c - f(a, b-1, c) = c$$
$$f(a, b-1, c) + a + b + c - f(a-1, b, c) = b + c.$$

Hence (3.2) reduces to

$$f(a, b, c) = 3(a+b+c) + f(a, b, c-1)$$

and it follows that (3.4) holds for $a+b+c = M$.

This completes the proof of (3.4).

The formulas (2.20) suggest the possibility of a relation of the following kind

$$F(a, b, c) = q^{d(a, b, c)}F(c, b, a), \tag{3.5}$$

for some integer $d(a, b, c)$. In view of (3.4)

$$d(a, b, c) = f(a, b, c) - f(c, b, a).$$

By (3.4) this gives

$$d(a, b, c) = (c-a)(a+b+c+1). \tag{3.6}$$

Thus (3.5) becomes

$$q^{a(N+1)}F(a, b, c) = q^{c(N+1)}F(c, b, a),\ N = a + b + c. \tag{3.7}$$

We shall show below (§5) by a combinatorial argument that (3.7) is indeed correct.

SECTION 4

Turning now to the general situation (1.6), we define the operators E_1^{-1}, E_2^{-1}, ..., E_n^{-1} by means of

$$E_j^{-1}\phi(\boldsymbol{e}) = \phi(\boldsymbol{e} - \delta_j), \tag{4.1}$$

where the notation is that in (1.5). We also put

$$\Omega = \sum_{j=1}^{n} q^{jN}E_j^{-1}\phi(\boldsymbol{e}) \qquad (N = e_1 + \cdots + e_n), \tag{4.2}$$

so that (1.6) becomes

$$F(\mathbf{e}) = \Omega F(\mathbf{e}) \qquad (N > 0). \tag{4.3}$$

Iteration of (4.3) gives

$$F(\mathbf{e}) = \Omega^m F(\mathbf{e}) \qquad (0 \le m \le N). \tag{4.4}$$

Generalizing (2.8), we write

$$\Omega^m = \sum_{\Sigma r_j = m} q^{\omega(\mathbf{r})N} C(\mathbf{r}) E_1^{-r_1} \ \dots \ E^{-r_n}, \tag{4.5}$$

where

$$= (r_1, r_2, \dots, r_n), \ \omega(r) = r_1 + 2r_2 + \cdots + nr_n \tag{4.6}$$

and $C(\mathbf{r})$ is independent of N. Then, in the first place, for $m = N$, (4.5) yields

$$F(\mathbf{e}) = q^{\omega(\mathbf{e})N} C(\mathbf{e}), \tag{4.7}$$

so that (3.5) becomes

$$\Omega^m = \sum_{\Sigma r_j = m} q^{\omega(\mathbf{r})(N-m)} C(\mathbf{r}) E_1^{-r_1} \ \dots \ E^{-r_n}. \tag{4.8}$$

It then follows from (4.5) and (4.8) that

$$F(\mathbf{e}) = \sum_{\Sigma r_j = m} q^{\omega(\mathbf{r})(N-m)} F(\mathbf{r}) F(\mathbf{e} - \mathbf{r}). \tag{4.9}$$

This result can be written in the more symmetrical form

$$F(\mathbf{e}) = \sum_{\substack{\Sigma r_j = m \\ r_j + r_j' = e_j}} q^{\omega(\mathbf{r})(N-m)} F(\mathbf{r}) F(\mathbf{r}'). \tag{4.10}$$

The remark about the equivalence of (2.13) and (2.14) is easily extended to (4.10).

As a simple application of (4.10) we take

$$F(a\delta_j) = q^{j(a-1)} F(\delta_j) F\big((a-1)\delta_j\big).$$

Then, since $F(\delta_j) = q^j$, we get

$$F(a\delta_j) = q^{\frac{1}{2}ja(a+1)} \qquad (1 \le j \le n). \tag{4.11}$$

This is evidently in agreement with (2.15).

Next

$$F(a, 1, 0, \dots, 0) = q^a F(1, 0, 0, \dots, 0) F(a-1, 1, 0, \dots, 0) \\ + q^{2a} F(0, 1, 0, \dots, 0) F(a, 0, 0, \dots),$$

which reduces to

$$F(a, 1, 0, \ldots, 0) = q^{a+1}F(a-1, 1, 0, \ldots) + q^{\frac{1}{2}a(a+1)+2a+2}.$$

This gives

$$F(a, 1, 0, \ldots, 0) = q^{\frac{1}{2}(a^2+3a+4)}(1 + q + \cdots + q^a). \tag{4.12}$$

For example

$$F(1, 1, 0, \ldots, 0) = q^4(1+q),\ F(2, 1, 0, \ldots, 0) = q^7(1+q+q^2),$$

in agreement with earlier results.

Clearly $F(\boldsymbol{e})$ is a polynomial in q with non-negative integral coefficients. Put

$$d(\boldsymbol{e}) = \deg F(\boldsymbol{e}), \tag{4.13}$$

the degree of $F(\boldsymbol{e})$. Then by (1.6)

$$d(\boldsymbol{e}) = \max_{1 \le j \le n} \left\{jN + d(\boldsymbol{e} - \delta_i)\right\}. \tag{4.14}$$

Thus, by (4.11)

$$\begin{aligned} d(\boldsymbol{e}) \ge n\big(N + (N-1) + \cdots + (N-e_n+1)\big) \\ + (n-1)\big((N-e_n) + (N-e_n-1) + \cdots + (N-e_n-e_{n-1}+1)\big) \\ + \cdots + 2\big((N-e_n-\cdots-e_3) + \cdots + (N-e_n-\cdots-e_2+1)\big) \\ + \frac{1}{2}e_1(e_1+1). \end{aligned}$$

After some manipulation this becomes

$$d(\boldsymbol{e}) \ge \frac{1}{2}N_1 + \frac{1}{2}N_2, \tag{4.15}$$

where

$$N_1 = \sum_{j=1}^{n} je,\quad N_2 = \sum_{i,j=1}^{n} \max(i, j)e_i e_j. \tag{4.16}$$

We shall show that indeed

$$d(\boldsymbol{e}) = \frac{1}{2}N_2 + \frac{1}{2}N_1. \tag{4.17}$$

To prove (4.17) it suffices to show that

$$N + d(\boldsymbol{e} - \delta_k) - d(\boldsymbol{e} - \delta_{k-1}) = e_k + e_{k+1} + \cdots + e_n \quad (k = 2, 3, \ldots, n)$$

under the assumption that (4.17) holds up to and including $N-1$. Making use of (4.17) we find that

$$d(\mathbf{e} - \delta_k) - d(\mathbf{e} - \delta_{k-1}) = \sum_{j=1}^{k-1} e_j \qquad (k = 2, 3, \ldots, n)$$

and (4.18) follows.

Corresponding to

$$\mathbf{e} = (e_1, e_2, \ldots, e_n)$$

we define

$$\mathbf{e}' = (e_n, e_{n-1}, \ldots, e_1).$$

Clearly

$$N = \sum_{j=1}^{n} e_j = \sum_{j=1}^{n} e_{n-j+1}.$$

However

$$N_1' = \sum_{j=1}^{n} je_{n-j+1} = \sum_{j=1}^{n} (n-j+1)e_j = (n+1)N - N_1.$$

Thus

$$d(\mathbf{e}') = \frac{1}{2}(N+1)\big((n+1)N - N_1\big), \tag{4.19}$$

so that

$$d(\mathbf{e}) - d(\mathbf{e}') = \frac{1}{2}(N+1)\big(2N_1 - (n+1)N\big). \tag{4.20}$$

In particulr, for $n = 3$, (4.20) reduces to

$$d(\mathbf{e}) - d(\mathbf{e}') = (N+1)(c-a)$$

in agreement with (3.6).

We shall show by a combinatorial argument in §5 that

$$F(\mathbf{e}) = q^{\frac{1}{2}(N+1)(2N_1-(n+1)N)} F(\mathbf{e}'). \tag{4.21}$$

SECTION 5

The combinatorial problem alluded to at the beginning of the paper is the following. Put

$$\mathbf{e} = (e_1, e_2, \ldots, e_n), \quad N = e_1 + e_2 + \cdots + e_n, \tag{5.1}$$

where the e_j are non-negative integers. Consider sequences of length N:

$$\sigma = (a_1, a_2, \ldots, a_N),$$

where the a_j are in $Z_n = \{1, 2, \ldots, n\}$ and each element i occurs exactly e_i times; $\mathbf{e}$ is called the *signature* of σ. We define the weight $\omega(\sigma)$ of σ by means of

$$\omega(\sigma) = \sum_{j=1}^{n} ja_j. \tag{5.3}$$

We seek $f(\boldsymbol{e}, k)$, the number of sequences σ from Z_n of signature $\boldsymbol{\sigma}$ and weight k. It is convenient to define a refinement of $f(\boldsymbol{e}, k)$. For $1 \le j \le n$, we let $f_j(\boldsymbol{e}, k)$ denote the number of sequences σ from Z_n of signature $\boldsymbol{\sigma}$, weight k, and with last element $a_N = j$. It follows immediately from the definition that

$$f(\boldsymbol{e}, k) = \sum_{j=1}^{n} f_j(\ , k). \tag{5.4}$$

Moreover

$$f_j(\boldsymbol{e}, k) = \sum_{i=1}^{n} f_i(\boldsymbol{e} - \delta_j, k - jN), \tag{5.5}$$

where δ_j has the same meaning as above.

Put

$$\begin{cases} F(\boldsymbol{e}) = F(\boldsymbol{e}, q) = \sum_k f(\boldsymbol{e}, k) q^k \\ F_j(\boldsymbol{e}) = F_j(\boldsymbol{e}, q) = \sum_k f_j(\boldsymbol{e}, k) q^k, \end{cases} \tag{5.6}$$

so that

$$F(\boldsymbol{e}) = \sum_{j=1}^{k} F_j(\boldsymbol{e}). \tag{5.7}$$

Multiplying both sides of (5.5) by q^k and summing over k, we get

$$\begin{aligned} F_j(\boldsymbol{e}) &= \sum_k \sum_{i=1}^{n} f_i(\boldsymbol{e} - \delta_j, k - jN) q^k \\ &= q^{jN} \sum_k f(\boldsymbol{e} - \delta_j, k) q^k \\ &= q^{jN} F(\boldsymbol{e} - \delta_j). \end{aligned}$$

Hence, summing over j, it is clear that

$$F(\boldsymbol{e}) = \sum_{j=1}^{n} q^{jN} F(\boldsymbol{e} - \delta_j). \tag{5.8}$$

This is identical with the recurrence (1.6); also $F(\boldsymbol{e})$ satisfies the same initial conditions as in §1.

The polynomial $F(\boldsymbol{e})$ also satisfies a second recurrence. To find this recurrence we let $\overline{f}_j(\boldsymbol{e}, k)$ denote the number of sequences σ from Z_n with signature $\boldsymbol{e}$, weight k, and *first* element $e_1 = j$. Then of course

$$f(\boldsymbol{e}, k) = \sum_{j=1}^{n} \bar{f}_j(\boldsymbol{e}, k). \tag{5.9}$$

We have also

$$\bar{f}_j(\boldsymbol{e}, k) = \sum_{i=1}^{n} \bar{f}_i(\boldsymbol{e} - \delta_j,\ k - N_1 + j) = f(\boldsymbol{e} - \delta_j,\ k - N_1 + j), \tag{5.10}$$

where

$$N_1 = e_1 + 2e_2 + \cdots + ne_n. \tag{5.11}$$

Hence, by (5.9)

$$F(\boldsymbol{e}) = \sum_{j=1}^{n} F(\boldsymbol{e} - \delta_j)q^{N_1 - j}.$$

Now put

$$\boldsymbol{e}' = (e_n,\ e_{n-1},\ \ldots,\ e_1) \tag{5.12}$$

and

$$\sigma' = (a'_N,\ a'_{N-1},\ \ldots,\ a'_1), \tag{5.13}$$

where

$$a'_j = n - a_j + 1 \qquad (j = 1,\ 2,\ \ldots,\ N).$$

Corresponding to (5.11), we put

$$N'_1 = e_n + 2e_{n-1} + \cdots + ne_1. \tag{5.14}$$

Thus

$$N_1 + N'_1 = (n+1)N. \tag{5.15}$$

The weight of σ' is evidently

$$\begin{aligned}\omega(\sigma') &= \sum_{j=1}^{N} ja'_{N-j+1} = \sum_{j=1}^{N}(N-j+1)a'_j = \sum_{j=1}^{N}(N-j+1)(n-a_j+1)\\ &= (n+1)N(N+1) - \frac{1}{2}(n+1)N(N+1) - (N+1)\sum_{j=1}^{N}a_j + \sum_{j=1}^{N}ja_j.\end{aligned}$$

This gives

$$\omega(\sigma') = \frac{1}{2}(n+1)N(N+1) - (N+1)N_1 + \omega(\sigma). \tag{5.16}$$

Thus there is a 1-1 correspondence between sequences σ of signature $\boldsymbol{e}$ and weight k, and sequences σ' of signature $\boldsymbol{e}'$ and weight

$$\frac{1}{2}(N+1)\big((n+1)N - 2N_1\big) + k,$$

so that

$$f(\boldsymbol{e}, k) = f\left(\boldsymbol{e}, \frac{1}{2}(N+1)\left((n+1)N - 2N_1\right) + k\right).$$

This yields

$$F(\boldsymbol{e}) = q^{\frac{1}{2}(N+1)(2N_1 - (n+1)N)} F(\boldsymbol{e}'), \tag{5.17}$$

so that we have proved (4.21).

It is proved in (4.17) that

$$\deg F(\boldsymbol{e}) = \frac{1}{2}(N_1 + N_2), \tag{5.18}$$

which implies

$$f(\boldsymbol{e}, k) = 0 \qquad \left(k > \frac{1}{2}(N_1 + N_2)\right). \tag{5.19}$$

Also the proof of (4.17) gives

$$f\left(\boldsymbol{e}, \frac{1}{2}(N_1 + N_2)\right) = 1. \tag{5.20}$$

In the next place, define

$$\overline{\sigma} = (a_N, a_{N-1}, \ldots, a_1),$$

so that

$$\omega(\overline{\sigma}) = \sum_{j=1}^{N} j a_{N-j+1} = \sum_{j=1}^{N} (N - j + 1) a_j = (N+1)N_1 - \omega(\sigma). \tag{5.21}$$

It therefore follows from (5.19) and (5.20) that

$$f(\boldsymbol{e}, k) = \begin{cases} 1 & \left(k = NN_1 + \frac{1}{2}(N_1 - N_2)\right) \\ 0 & \left(k < NN_1 + \frac{1}{2}(N_1 - N_2)\right). \end{cases} \tag{5.22}$$

Thus

$$\begin{cases} \omega_{\max}(\sigma) = \frac{1}{2}(N_1 + N_2) \\ \omega_{\min}(\sigma) = NN_1 + \frac{1}{2}(N_1 - N_2). \end{cases} \tag{5.23}$$

Finally it is evident from (5.21) that

$$F(\boldsymbol{e}, q) = q^{(N+1)N_1} F(\boldsymbol{e}, q^{-1}), \tag{5.24}$$

where we are using the fuller notation, $F(\boldsymbol{e}, q) = F(\boldsymbol{e})$.

Put

$$f_n(N, k) = \sum_{e_1 + \cdots + e_n = N} f(\boldsymbol{e}, k),$$

so that $f_n(N, k)$ is the number of sequences from Z_n of length N and weight k. Also put

$$F_n(N, q) = \sum_k f_n(N, k)q^k.$$

Then it follows almost immediately from the definition of $f_n(N, k)$ that

$$F_n(N, q) = q^{\frac{1}{2}N(N+1)} \prod_{j=1}^{N} \frac{1 - q^{nj}}{1 - q^j}. \tag{5.25}$$

Indeed it suffices to observe that the right-hand side of (5.25) is equal to

$$\prod_{j=1}^{N} (q^j + q^{2j} + \cdots + q^{nj}).$$

A curious partition identity is implied by (5.25). Put

$$\prod_{j=1}^{N} (1 - q \quad)^{-1} = \sum_{m=0}^{\infty} p(m, N)q^m,$$

so that $p(m, N)$ is the number of partitions of m into parts $\leq N$. Now rewrite (5.25) in the form

$$q^{\frac{1}{2}N(N+1)} \sum_{k=0}^{\infty} p(k, N)q^k = \sum_{m=0}^{\infty} p(m, N)\, q^{mn} \sum_k f_n(N, k)q^k.$$

Then, equating coefficients of q^k, we get

$$p\left(k - \frac{1}{2}N(N+1)\right) = \sum_{mn \leq k} p(m, N) f_n(N, k \ - mn). \tag{5.26}$$

Another identity is obtained by replacing n by $2n$ in (5.25):

$$F_{2n}(N, q) = q^{\frac{1}{2}N(N+1)} \prod_{j=1}^{N} \frac{1 - q^{2nj}}{1 - q^j}.$$

Then by division

$$F_{2n}(N, q) = F_n(N, q) \prod_{j=1}^{N} (1 + q^{nj}).$$

Hence, if we put

$$\prod_{j=1}^{N} (1 + q^j) = \sum_{m=0}^{\frac{1}{2}N(N+1)} \overline{p}(m, N)q^m,$$

so that $\overline{p}(m, N)$ is the number of partitions of m into distinct parts $\leq N$, we get

$$\sum_k f_{2n}(N, k)q^k = \sum_k f_n(N, k)q^k \sum_{m=0}^{\frac{1}{2}N(N+1)} \overline{p}(m, N)q^{mn}.$$

Therefore

$$f_{2n}(N, k) = \sum_{mn \leq k} \overline{p}(m, N) f_n(N, k - mn). \tag{5.27}$$

For references to other enumerative problems involving sequences see [1].

REFERENCE

[1] L. Carlitz, "Permutations, Sequences and Special Functions," *SIAM Review* Vol. 17 (1975), pp. 298-322.

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