THE NUMBER OF DERANGEMENTS OF A SEQUENCE WITH GIVEN SPECIFICATION

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SECTION 1

Consider sequences

$$\sigma = (a_1, a_2, \ldots, a_N), \tag{1.1}$$

where $a_j \in Z_k = \{1, 2, \ldots, k\}$. The sequence is said to have *specification* $[n_1, n_2, \ldots, n_k]$, where the n_j are non-negative integers, $N = n_1 + n_2 + \cdots + n_k$, if each element j, $1 \le j \le k$, occurs in σ exactly n_j times. The sequence is called a *derangement* provided no element is in a position occupied by it in the sequence

$$(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, k, k, \ldots, k). \tag{1.2}$$

Let $P(n_1, n_2, \ldots, n_k)$ denote the number of possible derangements. Even and Gil is [1] (see also Jackson [2]) have proved the following result.

$$P(n_1, n_2, \ldots, n_k) = (-1)^{n_1+n_2+\cdots+n_k} \cdot \int_0^\infty e^{-x}\left\{\prod_{j=1}^{k} L_{n_j}(x)\right\} dx, \tag{1.3}$$

where $L_n(x)$ is the Laguerre polynomial defined by

$$L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\frac{x^j}{j!}. \tag{1.4}$$

The object of the present note is to give a simple proof of (1.3) along the lines of the standard proof of the case $n_1 = n_2 = \cdots = n_k = 1$ [3, p. 59]. We also prove some related results.

SECTION 2

Let $$P(\boldsymbol{n}, \boldsymbol{m}) = P(n_1, \ldots, n_k; m_1, \ldots, m_k), \tag{2.1}$$

where $0 \le m_j \le n_j$, denote the number of sequences (1.1) in which, for each j, exactly m_j of the values remain in their original position in (1.2). It follows at once from the definition that

$$P(\boldsymbol{n}, \boldsymbol{m}) = P(\boldsymbol{n} - \boldsymbol{m}, \boldsymbol{0})\prod_{j=1}^{k}\binom{n_j}{m_j} = P(\boldsymbol{n} - \boldsymbol{m})\prod_{j=1}^{k}\binom{n_j}{m_j}, \tag{2.2}$$

where $P(\boldsymbol{n}) = P(n_1, n_2, \ldots, n_k)$.

Clearly

$$\sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} P(\boldsymbol{n}, \boldsymbol{m}) = (n_1, n_2, \ldots, n_k) = \frac{N!}{n_1!n_2! \ldots n_k!},$$

where

$$\sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} \equiv \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \cdots \sum_{m_k=0}^{n_k}.$$

Thus, by (2.2),

$$\sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} P(m) = (n_1, n_2, \ldots, n_k).$$

This relation is equivalent to

$$P(\boldsymbol{n}) = \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^{N-M} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} (m_1, \ldots, m_k) \tag{2.3}$$

$$= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^{N-M} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{M!}{m_1! \ldots m_k!},$$

where $M = m_1 + \cdots + m_k$.

SECTION 3

To verify that (2.3) is in agreement with (1.3), we take

$$\int_0^\infty e^{-x} \left\{ \prod_{j=1}^{k} L_{n_j}(x) \right\} dx = \int_0^\infty e^{-x} \left\{ \prod_{j=1}^{k} \sum_{m=0}^{n_j} (-1)^{m_j} \binom{n_j}{m_j} \frac{x^{m_j}}{m_j!} \right\} dx$$

$$= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^M \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{1}{m_1! \ldots m_k!} \int_0^\infty e^{-x} x^M dx$$

$$= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^M \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{M!}{m_1! \ldots m_k!}.$$

This evidently proves the equivalence of (1.3) and (2.3).

SECTION 4

Put

$$P_k(N) = \sum_{n_1 + \cdots + n_k = N} P(\boldsymbol{n}). \tag{4.1}$$

Thus $P(n)$ denotes the total number of derangements from Z_k of length N. Then by (2.3) we have

$$P_k(n) = \sum_{n_1 + \cdots + n_k = N} \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^{N-M} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{M!}{m_1! \ldots m_k!}$$

$$= \sum_{m_1 + \cdots + m_k = N} (-1)^{N-M} \frac{M!}{m_1! \ \ldots \ m_k!} \sum_{n_1 + \cdots + n_k = N} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k},$$

where as above $M = m_1 + \cdots + m_k$. Since the inner sum on the extreme right is equal to

$$\binom{N + k - 1}{M + k - 1},$$

we get

$$P_k(N) = \sum_{m_1 + \cdots + m_k \leq N} (-1)^{N-M} \frac{M!}{m_1! \ \ldots \ m_k!} \binom{N + k - 1}{M + k - 1}$$

$$= \sum_{M=0}^{N} (-1)^{N-M} \binom{N + k - 1}{M + k - 1} \sum_{m_1 + \cdots + m_k = M} \frac{M!}{m_1! \ \ldots \ m_k!}.$$

By the multinomial theorem

$$\sum_{m_1 + \cdots + m_k = M} \frac{M!}{m_1! \ \ldots \ m_k!} = k^M,$$

so that

$$P_k(N) = \sum_{M=0}^{N} (-1)^{N-M} \binom{N + k - 1}{M + k - 1} k^M \tag{4.2}$$

It follows from (4.2) that

$$k^{k-1} P_k(N) = \sum_{m=k-1}^{N+k-1} (-1)^{N+k-m-1} \binom{N + k - 1}{m} k^m$$

$$= \sum_{m=0}^{N+k-1} (-1)^{N+k-m-1} \binom{N + k - 1}{m} k^m - \sum_{j=0}^{k-2} (-1)^{N+k-j-1} \binom{N + k - 1}{j} k^j$$

and therefore

$$P_k(N) = k^{1-k} \left\{ (k - 1)^{N+k-1} - \sum_{j=0}^{k-2} (-1)^{N+k-j-1} \binom{N + k - 1}{j} k^j \right\} \qquad (k \geq 1). \tag{4.3}$$

It follows from (4.3) that, for fixed $k > 2$,

$$P_k(N) \sim k^{1-k} (k - 1)^{N+k-1} \qquad (N \to \infty). \tag{4.4}$$

On the other hand, if N is fixed and $k \to \infty$, it is evident from (4.2) that

$$P_k(N) = \sum_{M=0}^{N} (-1)^M \binom{N + k - 1}{M} k^{N-M} \sim \sum_{M=0}^{N} (-1)^M \frac{k^M}{M!} k^{N-M},$$

so that

$$P_k(N) \sim k^N \sum_{M=0}^{N} \frac{(-1)^M}{M!} \qquad (k \to \infty). \tag{4.5}$$

SECTION 5

Fairly simple generating functions are implied by (4.2). We have first

$$\begin{aligned}\sum_{N=0}^{\infty} x^N \sum_{M=0}^{N} (-1)^{N-M} \binom{N+k-1}{M+k-1} k^M &= \sum_{M=0}^{\infty} k^M x^M \sum_{N=0}^{\infty} (-1)^N \binom{N+M+k-1}{M+k-1} x^N \\ &= \sum_{M=0}^{\infty} k^M x^M (1+x)^{-M-k} \\ &= (1+x)^{-k}\left(1 - \frac{kx}{1+x}\right)^{-1}.\end{aligned}$$

Hence

$$\sum_{N=0}^{\infty} P_k(N) x^N = (1+x)^{-k+1}(1+x-kx)^{-1}. \tag{5.1}$$

In the next place

$$\begin{aligned}\sum_{N=0}^{\infty} P_k(N) \frac{x^N}{(N+k-1)!} &= \sum_{N=0}^{\infty} \frac{x^N}{(N+k-1)!} \sum_{M=0}^{N} (-1)^{N-M} \binom{N+k-1}{M+k-1} k^M \\ &= \sum_{M=0}^{\infty} \frac{k^M x^M}{(M+k-1)!} \sum_{N=0}^{\infty} (-1)^N \frac{x^N}{N!} \\ &= e^{-x} \sum_{M=0}^{\infty} \frac{k^M x^M}{(M+k-1)!}.\end{aligned}$$

Thus

$$\sum_{N=0}^{\infty} P_k(N) \frac{x^N}{(N+k-1)!} = (kx)^{-k+1} e^{-x} \left\{ e^{kx} - \sum_{m=0}^{k-2} \frac{k^m x^m}{m} \right\} \qquad (k \geq 1). \tag{5.2}$$

It is easily seen that (4.3) is implied by (5.2).

REFERENCES

[1] S. Even & J. Gillis, "Derangements and Laguerre Polynomials," *Math. Proc. Camb. Phil. Soc.*, Vol. 79 (1976), pp. 135-143.

[2] D. M. Jackson, "Laguerre Polynomials and Derangements," *Math. Proc. Camb. Phil. Soc.*, Vol. 80 (1976), pp. 213-214.

[3] John Riordan, *An Introduction to Combinatorial Analysis* (New York: John Wiley & Sons, Inc., 1958).

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