TOPOLOGICAL, MEASURE THEORETIC AND ANALYTIC PROPERTIES OF THE FIBONACCI NUMBERS

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Our purpose here is to develop a connection between the arithmetic, topological, measure theoretic and analytic properties of the set, F, of Fibonacci numbers. We begin by topologizing the set, Z, of integers in a rather natural way and then showing that F has a certain closure property.

Definition: Let $\mathcal{A}$ be the topology on Z generated by the set of all arithmetic progressions. That is, for any integer b, a neighborhood base at b is given by

$$\left\{\{an + b \mid n \in Z\} \mid a \in Z,\ a \neq 0\right\}.$$

Our main result is

Theorem 1: $\{0\} \cup F \cup -F$ is $\mathcal{A}$-closed.

Proof: We use the theorem of Halton [2] which states that if f_n is divided by f_m $(n > m)$ then either the remainder r is a Fibonacci number or $f_m - r$ is a Fibonacci number. Here f_n is the nth Fibonacci number.

Thus, if $r > 0$ and $r \notin F$, choose $f \in F$ such that $f > r$ and $f - r \notin F$. It is easy to check, using Halton's theorem, that

$$(*) \qquad \left(\{0\} \cup F \cap -F\right) \cap \left\{fn + r \mid n \in Z\right\} = \emptyset.$$

Also, if $r < 0$ and $-r \notin F$, choose $f \in F$ such that $f > -r$ and $f + r \notin F$. Again it follows that (*) holds. This establishes the result.

We will, in what follows, omit details for the sake of brevity. However, we cite references for those readers interested in the technicalities of the subject.

Observe that $\{0\} \cup F \cup -F$ is closed in any topology for Z which is finer than $\mathcal{A}$. Thus, $\{0\} \cup F \cup -F$ is (see [3, p. 87]) closed in Z with the relative Bohr topology. This allows us to deduce (since $\{0\} \cup F \cup -F$ is a Sidon set) that $\{0\} \cup F \cup -F$ is a strong Riesz set (see [3, p. 90]). Meyer has proved [3, p. 90] that the union of a strong Riesz set and a Riesz set is a Riesz set. One implication of this fact is the following extension of the F. and M. Riesz Theorem.

Corollary: Let T be the circle group (that is, the group, under multiplication, of complex numbers of modulus 1). Let μ be a bounded Borel measure on T. The nth Fourier-Stieltjes coefficient $\hat{\mu}(n)$ of μ is defined by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \qquad (n \in Z).$$

Suppose $\hat{\mu}(n) = 0$ for all $n > 0$ with $n \notin F$. Then μ is absolutely continuous with respect to Lebesgue measure on T. Indeed, if R is any Riesz set and $\hat{\mu}(n) = 0$ for all $n \notin R \cup F$, then μ is absolutely continuous with respect to Lebesgue measure on T.

Comments:

(i) Observe that since $f_{n+1}/f_n \to \frac{\sqrt{5}+1}{2}$, it follows that F is a Hadamard set (see [3]). The above Corollary holds for all Hadamard sets [3, p. 94], but the proof, which depends on the comparatively deep work of Strzelecki [4], is much more difficult. This difficulty is to be expected because it is easy to see that there are Hadamard sets H with H $\mathcal{A}$-dense in Z. Thus, it is the intrinsic arithmetical properties of F which enable us to give our proof of the Theorem.

(ii) For some interesting arithmetical examples of Riesz sets the reader is referred to [1].

It is well known that the closure in the Bohr compactification of any Hadamard set has Haar measure zero. Using Halton's theorem, we can give a simple proof of this result for the Fibonacci numbers.

Theorem 2: Let $\overline{F}$ denote the closure of F in the Bohr compactification of Z. Then $\mu(\overline{F}) = 0$ where μ is the Haar measure of the Bohr compactification of Z.

Proof: We shall use the elementary fact that Haar measure of the closure of an arithmetic progression in the Bohr compactification of Z is precisely the natural density of the progression. Thus, it suffices, given $\varepsilon > 0$, to imbed F in a union of residue classes, modulo some fixed modulus, such that the density of the union is less than ε.

Choose m so large that $\frac{2m}{f_m} < \varepsilon$, which can be done because F has density zero. For any n, by Halton's theorem, $f_n \in f_m Z + r$ where $0 \leq r < f_m$ and $r \in F$ or $f_m - r \in F$.

But, there are clearly at most $2m$ integers, r, with $0 \leq r < f_m$ and $r \in F$ or $f_m - r \in F$. Thus, F can be imbedded in a union of residue classes (mod f_m) whose density is $\frac{2m}{f_m} < \varepsilon$. This completes the proof.

REFERENCES

[1] R. E. Dressler & Louis Pigno, "On Strong Riesz Sets," *Colloq. Math.*, Vol. 29 (1974), pp. 157-158.

[2] J. H. Halton, "Fibonacci Residues," *The Fibonacci Quarterly*, Vol. 2 (1964), pp. 217-218.

[3] Y. Meyer, "Spectres des mesures et mesures absolument continues," *Studia Math.*, Vol. 30 (1968), pp. 87-99.
[4] E. Strzelecki, "A Problem on Interpolation by Periodic and Almost Periodic Functions," *Colloq. Math.*, Vol. 11 (1963), pp. 91-99.

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