EXPANSION OF THE FIBONACCI NUMBER F_{nm} IN nTH POWERS OF FIBONACCI OR LUCAS NUMBERS

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Fibonacci numbers are defined by the recurrence relation $F_m + F_{m+1} = F_{m+2}$ and the initial values $F_0 = 0$, $F_1 = 1$. Lucas numbers are defined by $L_m = F_{m-1} + F_{m+1}$. The well-known identities $F_{2m} = F_{m+1}^2 - F_{m-1}^2$ and $F_{3m} = F_{m+1}^3 + F_m^3 - F_{m-1}^3$ are shown to be the first members of two families of identities of a more general nature. Similar identities for L_{2m} and L_{3m} can be generalized in similar ways.

1. Let $n = 2p$ be an even positive integer, m be any integer, and k be any integer except zero. Then

$$F_{nm} = \sum_{r=-p}^{p} a_r F_{m+rk}^n = 5^{-p} \sum_{r=-p}^{p} a_r L_{m+rk}^n$$

where $a_0 = 0$, $a_{-r} = -a_r$ and $a_1, a_2, \ldots, a_p$ are the solution of the p simultaneous equations

$$(1) \quad \sum_{r=1}^{p} a_r (-1)^{rks} F_{rk(n-2s)} = \begin{cases} 5^{p-1} & \text{for } s = 0 \\ 0 & \text{for } s = 1, 2, \ldots, p-1 \end{cases}$$

2. Let n, p, m, and k be as in 1. Then

$$L_{nm} = \sum_{r=-p}^{p} b_r F_{m+rk}^n = 5^{-p} \sum_{r=-p}^{p} b_r L_{m+rk}^n$$

where $b_{-r} = b_r$ and $b_0, b_1, \ldots, b_p$ are the solution of the $p+1$ simultaneous equations

$$(2) \quad b_0 + \sum_{r=1}^{p} b_r (-1)^{rks} L_{rk(n-2s)} = \begin{cases} 5^p & \text{for } s = 0 \\ 0 & \text{for } s = 1, 2, \ldots, p. \end{cases}$$

3. Let $n = 2p+1$ be an odd positive integer, and let m and k be as in 1. Then

$$F_{nm} = \sum_{r=-p}^{p} c_r F_{m+rk}^n \quad \text{and} \quad L_{nm} = 5^{-p} \sum_{r=-p}^{p} c_r L_{m+rk}^n$$

where $c_{-r} = (-1)^{rk} c_r$ and $c_r = b_r$ for $r \geq 0$.

4. Since the proofs are similar in all cases, only that for the first identity need be given. The Fibonacci numbers are first written in the Binet form

$$F_u = 5^{-\frac{1}{2}}\{g^u - (-g)^{-u}\}$$

where $g = \frac{1}{2}\left(5^{\frac{1}{2}} + 1\right)$. Then for n even:

$$F_{nm} = 5^{-\frac{1}{2}}(g^{nm} - g^{-nm}) = \sum_{r=-p}^{p} a_r 5^{-\frac{1}{2}n}\{g^{m+rk} - (-g)^{-m-rk}\}^n$$

$$= 5^{-p}\sum_{s=0}^{n}\binom{n}{s}(-1)^{ms+s} g^{m(n-2s)} \sum_{r=-p}^{p} a_r(-1)^{rks} g^{rk(n-2s)}$$

Equating coefficients of like powers of g for each value of s gives:

(3) $5^{p-\frac{1}{2}} = \sum_{r=-p}^{p} a_r g^{rkn}$ for $s = 0$

(4) $5^{p-\frac{1}{2}} = -\sum_{r=-p}^{p} a_r g^{-rkn}$ for $s = n$

(5) $0 = \sum_{r=-p}^{p} a_r(-1)^{rks} g^{rk(n-2s)}$ for $s = 1, 2, \ldots, n-1$

These $n+1$ equations can be rewritten in terms of Fibonacci numbers as follows: Equating the coefficients of like powers of g in (3) and (4) gives $a_{-r} = -a_r$ and $a_0 = 0$. Equations (3) and (4) are thus equivalent and can be rewritten in a common form

(6) $$5^{p-1} = 5^{-\frac{1}{2}}\left\{\sum_{r=1}^{p} a_r g^{rkn} + \sum_{r=-1}^{-p} a_r g^{rkn}\right\} = 5^{-\frac{1}{2}}\sum_{r=1}^{p} a_r(g^{rkn} - g^{-rkn})$$

$$= \sum_{r=1}^{p} a_r F_{rkn}$$

Similarly, (5) can be rewritten as

(7) $0 = \sum_{r=1}^{p} a_r(-1)^{rks} F_{rk(n-2s)}$ for $s = 1, 2, \ldots, n-1$

However, since $F_{-u} = -(-1)^u F_u$, this summation is unchanged when s is replaced with $n-s$, and since each term is zero when $s = p$, only $\frac{1}{2}(n-2)$ $p-1$ values of s give independent equations. These values can be taken as $s = 1, 2, \ldots, p-1$. Thus (6) and (7) together give p equations for the coefficients $a_1, a_2, \ldots, a_p$, and it is obvious that the conditions for the existence and uniqueness of the solution are satisfied.

5. For small values of n, the explicit expressions for a_r and b_r obtained by solving (1) and (2) can be reduced to simple forms by repeated use of the identities $L_{2u} = L_u^2 - 2(-1)^u = 5F_u^2 + 2(-1)^u$. The results for $n = 2, 3, 4$ are:

$n = 2 \qquad 1/a_1 = F_2 \qquad 1/b_0 = -\tfrac{1}{2}(-1)^k F_k^2 \qquad 1/b_1 = F_k^2$

$n = 3 \qquad 1/b_0 = -(-1)^k F_k^2 \qquad 1/b_1 = L_k F_k^2$

$n = 4 \qquad 1/a_1 = F_{2k}\left\{F_k^2 - (-1)^k F_{2k}^2\right\} \qquad 1/a_2 = -(-1)^k L_{2k}/a_1$

$1/b_0 = \tfrac{1}{2}(-1)^k F_k^4 L_k^2 \qquad 1/b_1 = F_k/L_k a_1$

$1/b_2 = -(-1)^k L_k^2/b_1$

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