EXPANSION OF THE FIBONACCI NUMBER \( F_n \) IN \( n \)TH POWERS OF FIBONACCI OR LUCAS NUMBERS

A. S. GLADWIN
McMaster University, Hamilton, Ontario, Canada L854L7

Fibonacci numbers are defined by the recurrence relation \( F_n + F_{n+1} = F_{n+2} \) and the initial values \( F_0 = 0, F_1 = 1 \). Lucas numbers are defined by \( L_n = F_{n-1} + F_{n+1} \). The well-known identities \( F_{2m} = F_{m+1}^2 - F_m^2 \) and \( F_{3m} = F_{m+1}^3 + F_m^3 - F_{m-1}^3 \) are shown to be the first members of two families of identities of a more general nature. Similar identities for \( L_{2m} \) and \( L_{3m} \) can be generalized in similar ways.

1. Let \( n = 2p \) be an even positive integer, \( m \) be any integer, and \( k \) be any integer except zero. Then

\[
F_{nm} = \sum_{r=-p}^{p} a_r F_m^r = 5^{-p} \sum_{r=-p}^{p} a_r F_m^{r+k}
\]

where \( a_0 = 0 \), \( a_{-r} = -a_r \) and \( a_1, a_2, \ldots, a_p \) are the solution of the \( p \) simultaneous equations

\[
(1) \sum_{r=1}^{p} a_r (-1)^{rk} F_{mk(n-2s)} = \begin{cases} 
5^{p-1} & \text{for } s = 0 \\
0 & \text{for } s = 1, 2, \ldots, p - 1 
\end{cases}
\]

2. Let \( n, p, m, \text{ and } k \) be as in 1. Then

\[
L_{nm} = \sum_{r=-p}^{p} b_r L_m^r = 5^{-p} \sum_{r=-p}^{p} b_r L_m^{r+k}
\]

where \( b_r = b_r \) and \( b_0, b_1, \ldots, b_p \) are the solution of the \( p+1 \) simultaneous equations

\[
(2) b_0 + \sum_{r=1}^{p} b_r (-1)^{rk} L_{mk(n-2s)} = \begin{cases} 
5^p & \text{for } s = 0 \\
0 & \text{for } s = 1, 2, \ldots, p.
\end{cases}
\]

3. Let \( n = 2p + 1 \) be an odd positive integer, and let \( m \) and \( k \) be as in 1. Then

\[
F_{nm} = \sum_{r=-p}^{p} a_r F_m^r \quad \text{and} \quad L_{nm} = 5^{-p} \sum_{r=-p}^{p} a_r L_m^{r+k}
\]

where \( a_{-r} = (-1)^k a_r \) and \( a_r = b_r \) for \( r \geq 0 \).

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4. Since the proofs are similar in all cases, only that for the first identity need be given. The Fibonacci numbers are first written in the Binet form

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)$$

where $\varphi = \frac{1}{2} \left( \frac{5}{2} + 1 \right)$. Then for $n$ even:

$$F_{n} = 5^{-\frac{1}{2}} (g^n - g^{-n}) = \sum_{r=p}^{\infty} a_r 5^{-\frac{1}{2}} g^{n+r} - (-g)^{-n-r} a_r n$$

$$= 5^{-\frac{1}{2}} \sum_{s=0}^{n} (\frac{n}{s}) (-1)^{s} g^{m+s} a_m a_{n-2s} \sum_{r=p}^{\infty} a_r (-1)^{r} g^{r(n-2s)}$$

Equating coefficients of like powers of $g$ for each value of $s$ gives:

(3) $5^{n-\frac{1}{2}} = \sum_{r=p}^{\infty} a_r g^{r} n$ for $s = 0$

(4) $5^{n-\frac{1}{2}} = -\sum_{r=p}^{\infty} a_r g^{-r} n$ for $s = n$

(5) $0 = \sum_{r=p}^{\infty} a_r (-1)^{r} g^{r(n-2s)}$ for $s = 1, 2, \ldots, n - 1$

These $n+1$ equations can be rewritten in terms of Fibonacci numbers as follows: Equating the coefficients of like powers of $g$ in (3) and (4) gives $a_r = -a_r$ and $a_0 = 0$. Equations (3) and (4) are thus equivalent and can be rewritten in a common form

(6) $5^{n-\frac{1}{2}} = 5^{-\frac{1}{2}} \left( \sum_{r=p}^{\infty} a_r g^{r} n + \sum_{r=p+1}^{\infty} a_r g^{-r} n \right) = 5^{-\frac{1}{2}} \sum_{r=p}^{\infty} a_r (g^{r} n - g^{-r} n)$

$$= \sum_{r=p}^{\infty} a_r F_{r(n)}$$

Similarly, (5) can be rewritten as

(7) $0 = \sum_{r=p}^{\infty} a_r (-1)^{r} g^{r(n-2s)}$ for $s = 1, 2, \ldots, n - 1$

However, since $F_{-n} = (-1)^n F_n$, this summation is unchanged when $s$ is replaced with $n-s$, and since each term is zero when $s = p$, only $\frac{n}{2}(n-2)$ $p-1$ values of $s$ give independent equations. These values can be taken as $s = 1, 2, \ldots, p-1$. Thus (6) and (7) together give $p$ equations for the coefficients $a_1, a_2, \ldots, a_p$, and it is obvious that the conditions for the existence and uniqueness of the solution are satisfied.

5. For small values of $n$, the explicit expressions for $a_r$ and $b_r$ obtained by solving (1) and (2) can be reduced to simple forms by repeated use of the identities $L_{2u} = L_u^2 - 2(-1)^u = 5F_u^2 + 2(-1)^u$. The results for $n = 2, 3, 4$ are:
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\[
n = 2 \quad \frac{1}{a_1} = F_2 \quad \frac{1}{b_0} = -\frac{1}{2}(-1)^k F_k^2 \quad \frac{1}{b_1} = F_k^2
\]

\[
n = 3 \quad \frac{1}{b_0} = -(-1)^k F_k^2 \quad \frac{1}{b_1} = L_k F_k^2
\]

\[
n = 4 \quad \frac{1}{a_1} = F_{2k} \left\{ F_k^2 - (-1)^k F_{2k}^2 \right\} \quad \frac{1}{a_2} = -(-1)^k L_{2k}/a_1
\]

\[
\frac{1}{b_0} = \frac{1}{2}(-1)^k F_k^2 L_k^2 \quad \frac{1}{b_1} = F_k / L_k a_1
\]

\[
\frac{1}{b_2} = -(-1)^k L_k^2 / b_1
\]

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