SIMPLIFIED PROOF OF A GREATEST INTEGER FUNCTION THEOREM

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The purpose of this paper is to give a simple proof of a result, due originally to Anaya and Crump [1], involving the greatest integer function $[\cdot]$. In the following, $a = \frac{1+\sqrt{5}}{2} \doteq 1.618$, $b = \frac{1-\sqrt{5}}{2} \doteq -0.618$, and $F_n = \frac{a^n - b^n}{\sqrt{5}}$ defines the nth Fibonacci number for $n \geq 1$.

Definition: Let δ be defined by $\delta = \frac{1}{2} - \frac{b^2}{\sqrt{5}} > 0$.

Lemma 1: For $n \geq 2$, $\delta \leq \frac{1}{2} \pm \frac{b^n}{\sqrt{5}}$.

Proof: Equivalent to $\frac{-b^2}{\sqrt{5}} \leq \frac{\pm b^n}{\sqrt{5}}$ or $\pm b^n \leq b^2$, which is clearly true for $n \geq 2$, since $|b| < 1$.

Lemma 2: For $n \geq 2$ and any γ satisfying $|\gamma| < \delta$,

$$\left[\frac{a^n}{\sqrt{5}} + \gamma + \frac{1}{2}\right] = F_n.$$

Proof: We must show $F_n < \frac{a^n}{\sqrt{5}} + \gamma + \frac{1}{2} < F_n + 1$, or using $F_n = \frac{a^n - b^n}{\sqrt{5}}$ and $-|\gamma| \leq \gamma \leq |\gamma|$, the required inequality will be true if

$$\frac{-b^n}{\sqrt{5}} \leq -|\gamma| + \frac{1}{2} \leq |\gamma| + \frac{1}{2} < \frac{-b^n}{\sqrt{5}} + 1.$$

The extreme left and right inequalities reduce to $|\gamma| \leq \frac{1}{2} + \frac{b^n}{\sqrt{5}}$ and $|\gamma| < \frac{1}{2} - \frac{b^n}{\sqrt{5}}$, respectively, both valid for $|\gamma| < \delta$ by Lemma 1.

Theorem 1: (Cf. [1]): For $n \geq 1$ and $1 \leq k < n$, $\left[a^k F_n + \frac{1}{2}\right] = F_{n+k}$.

Proof: $\left[a^k F_n + \frac{1}{2}\right] = \left[\frac{a^k(a^n - b^n)}{\sqrt{5}} + \frac{1}{2}\right] = \left[\frac{a^{k+n}}{\sqrt{5}} - \frac{(-1)^k b^{n-k}}{\sqrt{5}} + \frac{1}{2}\right]$ (using $ab = -1$) $= F_{n+k}$ by Lemma 2 since $\left|\frac{(-1)^{k+1} b^{n-k}}{\sqrt{5}}\right| \leq \frac{|b|}{\sqrt{5}} < \delta$.

Corollary 1: ([2], pp. 34-35): $F_{n+1} = \left[aF_n + \frac{1}{2}\right]$ for $n = 2, 3, 4, \ldots$

Proof: Take $k = 1$ in the theorem and note $1 = k < n$ for $n = 2, 3, 4, \ldots$

Corollary 2: (Cf. [3], p. 22): $\left[\frac{a^n}{\sqrt{5}} + \frac{1}{2}\right] = F_n$ for all $n \geq 1$.

Proof: Clearly true for $n = 1$, since $a \doteq 1.618$. For $n \geq 2$, the result follows from Lemma 2 with $\gamma = 0$.

Note: The case $k = n$ is not treated in Theorem 1, and in fact the result of the theorem fails for $n \geq 1$, $1 \leq k \leq n$ when $n = 1$ and $k = 1$, since

$$\left[aF_1 + \frac{1}{2}\right] = \left[a + \frac{1}{2}\right] = [2.118] = 2 \neq F_2 = 1$$

(thus the statement of the theorem in [1] requires modification). However, we can easily prove the following:

Theorem 2: Let $n \geq 2$ and $k = n$. Then $\left[a^n F_n + \frac{1}{2}\right] = F_{2n}$.

Proof: $\left[a^n F_n + \frac{1}{2}\right] = \left[\frac{a^n}{\sqrt{5}}(a^n - b^n) + \frac{1}{2}\right] = \left[\frac{a^{2n}}{\sqrt{5}} - \frac{(-1)^n}{\sqrt{5}} + \frac{1}{2}\right] = \left[\frac{a^{2n}}{\sqrt{5}} \pm \frac{1}{\sqrt{5}} + \frac{1}{2}\right]$

which will be F_{2n} $\left(\text{since } \frac{a^{2n}}{\sqrt{5}} > F_{2n} \text{ and } \pm\frac{1}{\sqrt{5}} + \frac{1}{2} > 0\right)$ if $\frac{a^{2n}}{\sqrt{5}} \pm \frac{1}{\sqrt{5}} + \frac{1}{2} < F_{2n} + 1$

or $\frac{-1}{2} \pm \frac{1}{\sqrt{5}} < \frac{-b^{2n}}{\sqrt{5}}$, an inequality which is easily verified for $n \geq 2$.

With both n and k unrestricted positive integers, we can also state two simple inequalities which depend on the fact that $[\cdot]$ is a nondecreasing function of its argument.

Corollary 3:

(i) For n even, $\left[a^k F_n + \frac{1}{2}\right] \leq F_{n+k}$ $(n \geq 1,\ k \geq 1)$

(ii) For n odd, $\left[a^k F_n + \frac{1}{2}\right] \geq F_{n+k}$ $(n \geq 1,\ k \geq 1)$.

Proof:

(i) With n even, $\frac{a^n}{\sqrt{5}} > F_n$ and $\left[a^k F_n + \frac{1}{2}\right] \leq \left[\frac{a^{k+n}}{\sqrt{5}} + \frac{1}{2}\right] = F_{n+k}$ by Cor. 2.

(ii) Similarly, n odd implies $\frac{a^n}{\sqrt{5}} < F_n$ and $\left[a^k F_n + \frac{1}{2}\right] \geq \left[\frac{a^{k+n}}{\sqrt{5}} + \frac{1}{2}\right] = F_{n+k}$ again by application of Cor. 2.

We may also obtain a similar result on Lucas numbers due to Carlitz [4] by an analogous approach (recall $L_n = a^n + b^n$ for $n \geq 1$).

Lemma 2: For all $n \geq 4$ and γ satisfying $|\gamma| \leq b^2$, $\left[a^n + \gamma + \frac{1}{2}\right] = L_n$.

Proof: We must show $L_n \leq a^n + \gamma + \frac{1}{2} < L_n + 1$, or, using $L_n = a^n + b^n$,

$b^n \le \gamma + \frac{1}{2} < b^n + 1$. Since $|\gamma| \le b^2$, the required inequality is satisfied if

$$b^n \le -b^2 + \frac{1}{2} < b^2 + \frac{1}{2} < b^n + 1.$$

But $b^n + b^2 < \frac{1}{2}$ and $b^n - b^2 > -\frac{1}{2}$ for $n \ge 4$, so the result follows.

Theorem 3: (Cf. [4]): For $k \ge 2$ and $n \ge k + 2$, $\left[\alpha^k L_n + \frac{1}{2}\right] = L_{n+k}$.

Proof: $\left[\alpha^k L_n + \frac{1}{2}\right] = \left[\alpha^k(\alpha^n + b^n) + \frac{1}{2}\right] = \left[\alpha^{n+k} + (-1)^k b^{n-k} + \frac{1}{2}\right] = L_{n+k}$ by Lemma 2, since $|(-1)^k b^{n-k}| \le b^2$ and $n + k \ge 4$.

Corollary 4: $\left[\alpha^n + \frac{1}{2}\right] = L_n$ for $n \ge 2$.

Proof: For $n \ge 4$, result is established by Lemma 2 on taking $\gamma = 0$. For $n = 2, 3$, a direct verification suffices. [Recall $\alpha^2 = \alpha + 1$, so that $\alpha^3 = (\alpha + 1)\alpha = 2\alpha + 1$]. The result is also immediate from the fact that

$$|\alpha^n - (\alpha^n + b^n)| = |b|^n < \frac{1}{2} \text{ for } n \ge 2,$$

which shows that L_n is the closest integer to α^n for $n \ge 2$. It then follows that

$$\left[\alpha^n + \frac{1}{2}\right] = L_n \text{ for } n \ge 2.$$

REFERENCES

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