ON SQUARE PSEUDO-FIBONACCI NUMBERS

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If the Fibonacci numbers are defined by

$$u_1 = u_2 = 1,\ u_{n+2} = u_{n+1} = u_n,$$

then J. H. E. Cohn [1] has shown that

$$u_1 = u_2 = 1 \quad \text{and} \quad u_{12} = 144$$

are the only square Fibonacci numbers.

If n is a positive integer, we shall call the numbers defined by

$$u_1 = 1,\ u_2 = 4,\ u_{n+2} = u_{n+1} + u_n \tag{1}$$

pseudo-Fibonacci numbers.

The object of this paper is to show that the only square pseudo-Fibonacci numbers are

$$u_1 = 1,\ u_2 = 4, \text{ and } u_4 = 9.$$

If we remove the restriction $n > 0$, we obtain exactly one more square,

$$u_{-8} = 81.$$

It can easily be shown that the general solution of the difference equation (1) is given by

$$u_n = \frac{7}{5 \cdot 2^n}(\alpha^n + \beta^n) - \frac{1}{5 \cdot 2^{n-1}}(\alpha^{n-1} + \beta^{n-1}), \tag{2}$$

where

$$\alpha = 1 + \sqrt{5},\ \beta = 1 - \sqrt{5},$$

and n is an integer. Let

$$\eta_r = \frac{\alpha^r + \beta^r}{2^r},\ \xi_r = \frac{\alpha^r - \beta^r}{2^r\sqrt{5}}.$$

Then we easily obtain the following relations:

$$u_n = \frac{1}{5}(7\eta_n - \eta_{n-1}), \tag{3}$$

$$\eta_r = \eta_{r-1} + \eta_{r-2},\ \eta_1 = 1,\ \eta_2 = 3, \tag{4}$$

$$\xi_r = \xi_{r-1} + \xi_{r-2},\ \xi_1 = 1,\ \xi_2 = 1, \tag{5}$$

$$\eta_r^2 - 5\xi_r^2 = (-1)^r 4, \tag{6}$$

$$\eta_{2r} = \eta_r^2 + (-1)^{r+1}2, \tag{7}$$

$$2\eta_{m+n} = 5\xi_m\xi_n + \eta_m\eta_n, \tag{8}$$

$$2\xi_{m+n} = \eta_n\xi_m + \eta_m\xi_n, \tag{9}$$

$$\xi_{2r} = \eta_r\xi_r. \tag{10}$$

The following congruences hold:

$$u_{n+2r} \equiv (-1)^{r+1}u_n (\text{mod } \eta_r 2^{-S}), \tag{11}$$

$$u_{n+2r} \equiv (-1)^r u_n (\text{mod } \xi_r 2^{-S}), \tag{12}$$

where $S = 0$ or 1.

Let $\phi_t = \eta_{2^t}$, where t is a positive integer. Then we get

$$\phi_{t+1} = \phi_t^2 - 2. \tag{13}$$

We also need the following results concerning ϕ_t:

$$\phi_t \text{ is an odd integer}, \tag{14}$$

$$\phi_t \equiv 3 \pmod 4, \tag{15}$$

$$\phi_t \equiv 2 \pmod 3,\ t \geq 3. \tag{16}$$

We also have the following tables of values:

n	-8	0	1	2	3	4	5	7	9	11	12	13	15
u_n	81	3	1	4	5	9	14	37	97	254	411	665	1741

t	7	14
η_t	29	$3 \cdot 281$

t	4	7	8
ξ_t	3	13	$3 \cdot 7$

Let

$$x^2 = u_n. \tag{17}$$

The proof is now accomplished in sixteen stages:

(a) (17) is impossible if $n \equiv 3 \pmod 8$. For, using (12) we find that

$$u_n \equiv u_3 (\text{mod } \xi_4) \equiv 5 \pmod 3.$$

Since $\left(\frac{5}{3}\right) = -1$, (17) is impossible.

(b) (17) is impossible if $n \equiv 5 \pmod 8$. For, using (12) we find that

$$u_n \equiv u_5 \pmod{\xi_4} \equiv 14 \pmod 3.$$

Since $\left(\frac{14}{3}\right) = -1$, (17) is impossible.

(c) (17) is impossible if $n \equiv 0 \pmod{16}$. For, using (12) in this case

$$u_n \equiv u_0 \pmod{\xi_8} \equiv 3 \pmod 7, \text{ since } 7 | \xi_8.$$

Since $\left(\frac{3}{7}\right) = -1$, (17) is impossible.

(d) (17) is impossible if $n \equiv 15 \pmod{16}$. For, using (12) we find that

$$u_n \equiv u_{15} \pmod{\xi_8} \equiv 1741 \pmod 7, \text{ since } 7 | \xi_8.$$

Since $\left(\frac{1741}{7}\right) = -1$, (17) is impossible.

(e) (17) is impossible if $n \equiv 12 \pmod{16}$. For, using (12) in this case

$$u_n \equiv u_{12} \pmod{\xi_8} \equiv 411 \pmod 7, \text{ since } 7 | \xi_8.$$

Since $\left(\frac{411}{7}\right) = -1$, (17) is impossible.

(f) (17) is impossible if $n \equiv 7 \pmod{14}$. For, using (12) we find that

$$u_n \equiv \pm u_7 \pmod{\xi_7} \equiv \pm 37 \pmod{13}.$$

Since $\left(\frac{-37}{13}\right) = \left(\frac{37}{13}\right) = -1$, (17) is impossible.

(g) (17) is impossible if $n \equiv 3 \pmod{14}$. For, using (12) in this case

$$u_n \equiv \pm u_3 \pmod{\xi_7} \equiv \pm 5 \pmod{13}.$$

Since $\left(\frac{-5}{13}\right) = \left(\frac{5}{13}\right) = -1$, (17) is impossible.

(h) (17) is impossible if $n \equiv 5 \pmod{14}$. For, using (11) we find that

$$u_n \equiv u_5 \pmod{\eta_7} \equiv 14 \pmod{29}.$$

Since $\left(\frac{14}{29}\right) = -1$, (17) is impossible.

(i) (17) is impossible if $n \equiv 13 \pmod{14}$. Sor, using (12) in this case

$$u_n \equiv \pm u_{13} \pmod{\xi_7} \equiv \pm 665 \pmod{13}.$$

Since $\left(\frac{-665}{13}\right) = \left(\frac{665}{13}\right) = -1$, (17) is impossible.

(j) (17) is impossible if $n \equiv 11 \pmod{14}$. For, using (12) we find that

$$u_n \equiv \pm u_{11} \pmod{\xi_7} \equiv \pm 254 \pmod{13}.$$

Since $\left(\frac{-254}{13}\right) = \left(\frac{254}{13}\right) = -1$, (17) is impossible.

(k) (17) is impossible if $n \equiv 9 \pmod{14}$. For, using (12) we find that

$$u_n \equiv \pm u_9 \pmod{\xi_7} \equiv \pm 97 \pmod{13}.$$

Since $\left(\frac{-97}{13}\right) = \left(\frac{97}{13}\right) = -1$, (17) is impossible.

(l) (17) is impossible if $n \equiv 15 \pmod{28}$. For, using (11) we find that

$$u_n \equiv \pm u_{14} \pmod{\eta_{14}} \equiv \pm 1741 \pmod{281}, \text{ since } 281/\eta_4.$$

Since $\left(\frac{-1741}{281}\right) = \left(\frac{1741}{281}\right) = -1$, (17) is impossible.

(m) (17) is impossible if $n \equiv 1 \pmod 4$, $n \neq 1$, that is, if $n = 1 + 2^t r$, where r is odd and t is a positive integer ≥ 2. For, using (11) in this case

$$u_n \equiv -u_1 \pmod{\eta_2 t - 1} \equiv -1 \pmod{\phi_{t-1}}.$$

Now, using (15) we have $\phi_{t-1} = 4k + 3$, where k is a nonnegative integer. Since $\left(\frac{-1}{\phi_{t-1}}\right) = \left(\frac{-1}{4k+3}\right) = -1$, (17) is impossible.

(n) (17) is impossible if $n \equiv 2 \pmod 4$, $n \neq 2$, that is, if $n = 2 + 2^t r$, where r is odd and t is a positive integer ≥ 2. For, using (11) we find that

$$u_n \equiv -u_2 \pmod{\eta_2 t - 1} \equiv -4 \pmod{\phi_{t-1}}.$$

Now, using (15) we have $\phi_{t-1} = 4k + 3$, where k is a nonnegative integer. By virtue of (14), $(2, \phi_{t-1}) = 1$. Since $\left(\frac{-4}{\phi_{t-1}}\right) = \left(\frac{-4}{4k+3}\right) = -1$, (17) is impossible.

(o) (17) is impossible if $n \equiv 4 \pmod{16}$, $n \neq 4$, that is, if $n = 4 + 2^t r$, where r is odd and t is a positive integer ≥ 4. For, using (11) we find that

$$u_n \equiv -u_4 \pmod{\eta_{2^{t-1}}} \equiv -9 \pmod{\phi_{t-1}}.$$

Now, using (16), we get $(\phi_{t-1}, 3) = 1$, and by virtue of (15), $\phi_{t-1} = 4k + 3$, where k is a positive integer ≥ 11.

Next, since $\left(\frac{-9}{\phi_{t-1}}\right) = \left(\frac{-9}{4k+3}\right) = -1$, (17) is impossible.

(p) (17) is impossible if $n \equiv -8 \pmod{16}$, $n \neq -8$, that is, if $n = -8 + 2^t r$, where r is odd and t is a positive integer ≥ 4. For, using (11) in this case

$$u_n \equiv -u_{-8} \pmod{\eta_2 t - 1} \equiv -81 \pmod{\phi_{t-1}}.$$

Now, using (16) we get $(\phi_{t-1}, 3) = 1$, and by virtue of (15), $\phi_{t-1} = 4k + 3$, where k is a positive integer ≥ 11.

Next, since $\left(\frac{-81}{\phi_{t-1}}\right) = \left(\frac{-81}{4k + 3}\right) = -1$, (17) is impossible.

We have now four further cases, $n = -8$, 1, 2, and 4, to consider.

(1) When $n = -8$, $u_n = 81$ is a perfect square.

(2) When $n = 1$, $u_n = 1$ is a perfect square.

(3) When $n = 2$, $u_n = 4$ is a perfect square.

(4) When $n = 4$, $u_n = 9$ is a perfect square.

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REFERENCE

1. J. H. E. Cohn, "On Square Fibonacci Numbers," *J. London Math. Soc.*, Vol. 39 (1964), pp. 537-540.

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