SOLVED, SEMI-SOLVED, AND UNSOLVED PROBLEMS IN GENERALIZED INTEGERS: A SURVEY

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1. DEFINITION AND DESCRIPTION OF GENERALIZED INTEGERS

The original definition of generalized integers and the name of "generalized primes" were given by Arne Beurling in 1937 (*Acta. Math.*, Vol. 68, pp. 255-291).* Translated from the French, the notation changed, and the word "finite" added, Beurling's definition was "With every sequence, finite or infinite, of real numbers $\{p\}$:

$$1 < p_1 < p_2 < \cdots < p_n < \cdots \tag{1}$$

we can associate a new sequence $\{g\}$:

$$1 = g_1 \leq g_2 \leq g_3 \leq \cdots \leq g_n \leq \cdots \tag{2}$$

formed by the set of products

$$g = p_{n_1} p_{n_2} \cdots p_{n_r}, \; n_1 \leq n_2 \leq \cdots \leq n_r, \; r \geq 1 \tag{3}$$

with the convention that $g_1 = 1$ and every other number g appears in (2) as many times as it has distinct representations (3). We call the p_n the generalized primes (g.p.) of the sequence $\{g\}$ and designate by $\pi(x)$ the number of $p_n \leq x$ and by $N(x)$ the number of $g_n \leq x$." It was Bertil Nyman (1949) who first used the term "generalized integer" (g.i.) to denote the numbers g_n and first referred to Beurling's paper, although V. Ramaswami (1943) seems to have independently invented generalized integers.

Thus the generalized primes need not be natural primes, nor even integers. Also, factorization of generalized integers need not be unique. From the definition, the basic properties of the g.i. are that they can be multiplied and ordered, that is, counted, but not added. The following three sequences all fit the definition of a sequence of generalized primes $\{p_n\}$ together with the corresponding sequence of generalized integers $\{g_n\}$.

$$\begin{aligned} \{p_n\} &= (2, 5, 11, \ldots) \\ \{g_n\} &= (1, 2, 4, 5, 8, 10, 11, \ldots) \end{aligned} \tag{4}$$

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*A bibliography of the work on generalized integers is given at the end of this paper.

$$\{p_n\} = \left(\frac{13}{11}, \frac{7}{5}, \frac{3}{2}\right)$$
$$\{g_n\} = \left(1, \frac{13}{11}, \frac{13^2}{11^2}, \frac{7}{5}, \frac{3}{2}, \frac{13^3}{11^3}, \frac{13}{11} \quad \frac{7}{5}, \ldots\right) \tag{5}$$

$$\{p_n\} = (2, 3, 4, 5, 7, 11, 13, \ldots)$$
$$\{g_n\} = (1, 2, 3, 2.2, 4, 5, 6, 7, 2.2.2, 2.4, 9, 10, 11, 2.2.3, 4.3, 13, \ldots). \tag{6}$$

If unique factorization is also assumed, then analogues of the well-known multiplicative arithmetical functions, for example the Moebius function, can be defined and theorems, such as the Moebius inversion formula for g.i., can be proved. Much of this work has been carried out by the author.

However, for his work, Beurling needed an assumption on the size of $N(x)$. He and later writers on this topic were mainly concerned with the way in which $N(x)$ affected $\pi(x)$ and vice versa.

It can be seen that taking the g.p. to be a subset of the natural primes also fits the definition, but this covers a very large block of the total work done in Number Theory. Thus, this work is only included when it has been used in the context of generalized integers, even when the numbers being studied are also ordinary integers.

2. HISTORY OF THE SOLVED PROBLEMS

As succinctly stated by Beurling, his original question was "in what manner should $\varepsilon(x)$ converge to zero when x tends to infinity, so that the hypothesis $N(x) = x(A + \varepsilon(x))$, A a positive constant, infers the asymptotic law $\pi(x) \sim x/\log x$?" In fact he showed that the hypothesis $N(x) = Ax + 0(x/\log^\gamma x)$, $p_n \to \infty$, implies $\pi(x) \sim x/\log x$ if $\gamma > 3/2$, but it can fail to hold if $\gamma \leq 3/2$. This was proved using a complex variable and a zeta function for the g.i. Thus, Beurling was concerned with having a prime number theorem (p.n.t.) for generalized integers. Much of the later work revolved about this topic mainly in the direction of refinements in the hypothesis on $N(x)$. Amitsur (1961) gave an elementary proof of the p.n.t. when $N(x) = Ax + 0(x/\log^\gamma x)$ holds with $\gamma > 2$.

B. M. Bredihin (1958-1967) used the following algebraic definition: "Let G be a free commutative semi-group with a countable system P of generators. Let N be a homomorphism of G onto a multiplicative semi-group of numbers such that, for a given number x, only finitely many elements α in G have norm $N(\alpha)$ satisfying $N(\alpha) \leq x$." It can be seen that Bredihin's definition includes Beurling's definition of the g.i. except that factorisation of any α is unique but more than one element can have the same norm. Bredihin used the hypothesis

$$N(x) = Ax^\theta + 0(x^{\theta_1}),\ \theta > 0, \text{ and } \theta_1 < \theta;$$

and proved that

$$\lim_{x \to \infty} \pi(x) \log x/x^\theta = 1/\theta.$$

He was the first (1958) to publish an elementary proof of a prime number theorem for g.i.

During the 1950's, A. E. Ingham gave lectures on Beurling's work in Cambridge, England, and thus extended interest in the g.i.

The motivation for Beurling's work was to find how the number of generalized integers affected the number of generalized primes, and later on the converse problem was studied. A history, bibliography up to 1966, and demonstration using complex variable of the work done on this problem up to 1969, excluding the work of Bredihin and Rémond (1966), is given in Bateman and Diamond's article in Volume 6, MAA *Studies in Number Theory* (1969). Again, "the additive structure of the positive integers is not particularly relevant to the distribution of primes. As the g.i. have no additive structure, they are particularly useful to examine the stability of the prime number theorem." Work on the error term for $\pi(x)$ and the converse problem has been carried out by Nyman (1949), Malliavin (1961), Diamond (1969, 1970), and later writers.

A more complete history of generalized integers may be obtained by reading the reviews which W. J. Le Veque has classified under N80 in his "Reviews in Number Theory" (1974).

Recently, the work on generalized integers has been given a completely different twist by J. Knopfmacher in a series of papers (1972) to (1975) and later papers and a book *Abstract Analytic Number Theory*, Volume 12 of the North-Holland Mathematical Library and published by North-Holland, Amsterdam (1975). This book contains a complete bibliography up to 1974. In essence, Dr. Knopfmacher has used the techniques associated with generalized integers to prove an abstract prime number theorem for an "arithmetical semigroup" and has applied it to contexts not previously considered by other writers.

Segal's 1974 paper has the descriptive title "Prime Number Theorem Analogues without Primes." He states that the underlying multiplicative structure for g.i. is not all-important—a growth function is all that is needed.

There are also papers combining Beurling's generalized integers with an analogue of primes in arithmetic progressions. This means that the ideas of multiplicative semi-groups without addition had to be combined with the idea of primes in arithmetic progressions. The possibility of such an effort was worked on by Fogels (1964-1966) and Rémond (1966). Knopfmacher has used an arithmetical semi-group to attack a similar problem.

3. UNSOLVED PROBLEMS

Most unsolved problems are associated with the prime number theorem for generalized integers. One such set by Bateman and Diamond is listed on pages 198-200 in Volume 6 of *Studies in Number Theory*, mentioned previously. Of these, one was solved by Diamond and R. S. Hall in 1973. Another states that Beurling's theorem can be established by elementary methods with $\gamma < 2$, but that no one has yet succeeded in establishing it by elementary methods for $3/2 < \gamma < 2$. Dr. Diamond has also published four further problems from the Séminaire de Théorie des Nombres, 1973-74. He conjectures that

$$\int_1^\infty |N(x) - x| x^{-2}\, dx < \infty \Rightarrow \psi(x) \ll x.$$

Segal's paper of 1974 concludes with some open questions (pages 21 and 22).

A list of very general open questions is posed by Dr. Knopfmacher on pages 287-292 of his book (1975).

4. SOLVED PROBLEMS—ARITHMETICAL FUNCTIONS

In my own work on generalized integers (1961 to 1968), I have assumed the g.i. to be not necessarily integers but with unique factorization. Some of my papers on g.i. have concentrated on their arithmetical properties, that is, without a hypothesis on $N(x)$, and it is those I am concerned with here. Those needing a hypothesis on $N(x)$, I assume included in §2. However, the fact that the g.i. can be ordered and so a counting function $N(x)$ exists, is important.

Since there will now be a slight change in notation, I will repeat the definition I shall use for generalized primes and integers.

Suppose, given a finite or infinite sequence of real numbers (generalized primes) such that

$$1 < p_1 < p_2 < p_3 < \dots .$$

Form the set $\{g\}$ of all possible -products, i.e., products $p_1^{v_1} p_2^{v_2} \dots$, where $v_1, v_2, \dots$ are integers ≥ 0 of which all but a finite number are 0. Call these numbers "generalized integers" and suppose that no two generalized integers are equal if their v's are different. Then arrange $\{g\}$ as an increasing sequence:

$$1 = g_1 < g_2 < g_3 < \dots .$$

Notice that the g.i. cannot be added to give another g.i. For example, in (4), $1 + 2 = 3$ but $3 \notin \{g_n\}$.

However, division of one g.i. by another is easily defined as follows: We say $d | g_n$ if $\exists\, D$ so that $dD = g_n$ and both d and D belong to $\{g\}$. From these definitions, it follows that greatest common divisor, multiplicative functions, Moebius function, Euler ϕ-function, unitary divisors, etc., for the g.i. can be defined. These lead to further arithmetical properties mainly published by the author.

H. Gutmann (1959) and H. Wegmann (1966) also published results on properties of arithmetical functions. Gutmann worked with g.i. which were subsets of the natural numbers and he assumed that both

$$\sum_1^\infty p_n^{-1} \text{ and } \sum_1^\infty p_n^{-1} \log p_n$$

were convergent.

5. A SEMI-SOLVED PROBLEM

Consider the sequence of natural numbers

$$1, 2, 3, \dots, d, \dots, n, \dots,$$

and let $[x]$ denote the number of integers $\leq x$. Then, since every dth number is divisible by d, it follows that $[x/d]$ will give the number of integers $\leq x$ which are also divisible by d, for they are the numbers

$$1 \cdot d, 2 \cdot d, 3 \cdot d, \dots, \left[\frac{x}{d}\right] \cdot d.$$

Hence, if $n = a + b$ so that $\frac{n}{d} = \frac{a}{d} + \frac{b}{d}$, it follows that $\left[\frac{n}{d}\right] \geq \left[\frac{a}{d}\right] + \left[\frac{b}{d}\right]$, and, in particular, if d is a prime p, then

$$\left[\frac{n}{p}\right] \geq \left[\frac{a}{p}\right] + \left[\frac{b}{p}\right].$$

We now extend these ideas to generalized integers with unique factorization. In order to show the similarity between the integral value function $[x]$ and the counting function $N(x)$ for generalized integers, we change the notation and define

$$[x] = \text{number of g.i.} \leq x = N(x). \quad \text{Then } [g_n] = n.$$

Again, if in the sequence $g_1, g_2, \ldots, g_n$, $d|g_n$, there are in this sequence $\left[\frac{g_n}{d}\right]$ multiples of d, namely the numbers

$$1 \cdot d,\ g_2 \cdot d,\ \ldots,\ g\quad \cdot d,\ \ldots,\ \left[\frac{g_n}{d}\right] \cdot d.$$

So $\left[\frac{x}{d}\right]$ will again give the number of g.i. $\leq x$ which are also divisible by d. Suppose now that $n = a + b$, i.e., $[g_n] = [g_a] + [g_b]$. Is it still true that $\left[\frac{g_n}{d}\right] \geq \left[\frac{g_a}{d}\right] + \left[\frac{g_b}{d}\right]$, and in particular $\left[\frac{g_n}{p}\right] \geq \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right]$, where p is a generalized prime?

Consider the following example:

$\{p\}$	$5 < 7 < 8 < 11 < 13 < 29 < \ldots$
$\{g\}$	$1 < 5 < 7 < 8 < 11 < 13 < 25 < 29 < \ldots$
$[x]$	1 2 3 4 5 6 7 8 ...

Then $[8] = 4 = 2 + 2 = [5] + [5]$.

Now take $p = 5$ and we have $\left[\frac{8}{5}\right] = 1$ and $\left[\frac{5}{5}\right] = 1$. So in this case

$$\left[\frac{g_n}{p}\right] = \left[\frac{8}{5}\right] = 1 \quad \text{and} \quad \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right] = \left[\frac{5}{5}\right] + \left[\frac{5}{5}\right] = 2$$

so that

$$\left[\frac{g_n}{p}\right] < \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right].$$

So far as I know, the way in which g is affected by always (or sometimes) having the generalized primes constructed so that

$$\left[\frac{g_n}{p}\right] < \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right] \quad \text{when} \quad [g_n] = [g_a] + [g_b]$$

has not been investigated.

We now consider the way in which g is affected by always having the generalized primes constructed so that

$$\left[\frac{g_n}{p}\right] \geq \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right] \quad \text{when} \quad [g_n] = [g_a] + [g_b]. \tag{7}$$

Given the sequence $p_1 < p_2 < p_3 < \ldots$, the sequence of g.i. must begin

$$1 < p_1 < p_2 < \cdots \tag{8}$$

or

$$1 < p_1 < p_1^2 < p_2 < \cdots \tag{9}$$

or

$$1 < p_1 < p_1^2 < p_1^3 < p_2 < \cdots, \text{ etc.} \tag{10}$$

Suppose we assume (9) to be the case; then

$$\begin{array}{lccccc} \{g\} & 1 < & p_1 < & p_1^2 < & p_2 < & \cdots \\ [x] & 1 & 2 & 3 & 4 & \cdots \end{array}$$

and g_5 has to be found.

First notice that if $\left[\dfrac{g_a}{p}\right] \geq [g_c]$, then $\dfrac{g_a}{p} \geq g_c$. Now

$$[g_5] = 5 = \begin{matrix} 1 + 4 \\ 2 + 3 \end{matrix} = \begin{matrix} [1] + [p_2] \\ [p_1] + [p_1^2] \end{matrix}.$$

So

$$\left[\frac{g_5}{p_1}\right] \geq \begin{matrix} \left[\dfrac{1}{p_1}\right] + \left[\dfrac{p_2}{p_1}\right] \\ \left[\dfrac{p_1}{p_1}\right] + \left[\dfrac{p_1^2}{p_1}\right] \end{matrix} = \begin{matrix} 0 + 2 \\ 1 + 2 \end{matrix} = 3 = [p_1^2].$$

Hence, $g_5 \geq p_1^3$, and since p_1^3 has not yet occurred, it follows that $g_5 = p_1^3$.

Repeating the process, we have

$$[g_6] = 6 = \begin{matrix} 1 + 5 \\ 2 + 4 \\ 3 + 3 \end{matrix} = \begin{matrix} [1] + [p_1^3] \\ [p_1] + [p_2] \\ [p_1^2] + [p_1^2] \end{matrix}.$$

Hence,

$$\left[\frac{g_6}{p_1}\right] \geq \begin{matrix} 0 + 3 \\ 1 + 2 \\ 2 + 2 \end{matrix} = 4 = [p_2].$$

Hence, $g_6 = p_1 p_2$.

$$[g_7] = 7 = \begin{matrix} 1 + 6 \\ 2 + 5 \\ 3 + 4 \end{matrix} = \begin{matrix} [1] + [p_1 p_2] \\ [p_1] + [p_1^3] \\ [p_1^2] + [p_2] \end{matrix}.$$

So

$$\left[\frac{g_7}{p_1}\right] \geq \begin{array}{c} 0+4 \\ 1+3 \\ 2+2 \end{array} = 4 = [p_2]$$

Hence $g_7 \geq p_1 p_2$, which we knew already. It is certainly true that

$$\left[\frac{g_7}{p_2}\right] \geq \left[\frac{p_1 p_2}{p_2}\right] = 2$$

is not less than any combination of $\left[\frac{g}{p_2}\right] + \left[\frac{g}{p_2}\right]$. The value for g_7 is therefore not precisely determined. If it is not to be p_3, since the sequence is now

$$\begin{array}{cccccc} 1 < & p_1 < & p_1^2 < & p_2 < & p_1^3 < & p_1 p_2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array},$$

then g_7 must be p_1^4. However, since there is room for another prime, and the assumption will still be satisfied, I take $g_7 = p_3$. Although we will not go through the routine, the next number must be p_1^4.

$$[g_8] = \begin{array}{ll} 1+7 & [1] + [p_3] \\ 2+6 & [p_1] + [p_1 p_2] \\ 3+5 & [p_1^2] + [p_1^3] \\ 4+4 & [p_2] + [p_2] \end{array}$$

So

$$\left[\frac{g_8}{p_1}\right] \begin{array}{c} 0+4 \\ 1+4 \\ 2+3 \\ 2+2 \end{array} = 5 = [p_1^3]$$

Hence $g_8 = p_1^4$, as stated previously. Notice that the maximum value of $\left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right]$ can only increase by 1 at each step in the routine.

If this process is continued with primes being inserted wherever there is room for them, what sequence is developed and is the inequality sufficient to determine it?

(a) If the sequence starts as (8), then the total sequence becomes the natural numbers when $p_1 = 2$ and $p_2 = 3$. This was proved by induction by the author and D. E. Daykin. Dr. Daykin also proved that a sequence generated by only two generalized primes always satisfies the inequality. He conjectured that a sequence generated by three generalized primes only cannot satisfy the inequality.

(b) In his Master's thesis for the University of Melbourne (1968), R. B. Eggleton solved a variant of the problem in an algebraic context. He also proved that if the sequence starts as (8) then the total sequence is isomorphic to the natural numbers under multiplication.

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THE NUMBER OF PRIMES IS INFINITE

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For the theorem used as the title of this paper, many proofs exist, some simple, some erudite. For earlier proofs, we refer to [1]. We present here three interesting proofs of the above theorem, and believe that they are new in some sense.

Theorem 1: Let $A_0 = \alpha + m$, where α and m are positive integers with $(\alpha, m) = 1$. Let A_n be defined recursively by

$$A_{n+1} = A_n^2 - mA_n + m.$$

Then each A_i is prime to every A_j, $j \neq i$.

Proof: By definition,

$$A_1 = A_0^2 - mA_0 + m = \alpha A_0 + m.$$

Again,

$$A_2 = A_1^2 - mA_1 + m = A_1(A_1 - m) + m = \alpha A_0 A_1 + m.$$