Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers \( F \) and Lucas numbers \( L \) satisfy \( F_{n+2} = F_{n+1} + F_n \), \( F_0 = 0, F_1 = 1 \) and \( L_{n+2} = L_{n+1} + L_n \), \( L_0 = 2, L_1 = 1 \). Also \( a \) and \( b \) designate the roots \( (1 + \sqrt{5})/2 \) and \( (1 - \sqrt{5})/2 \), respectively, of \( x^2 - x - 1 = 0 \). 

PROBLEMS PROPOSED IN THIS ISSUE

B-382 Proposed by A. G. Shannon, N.S.W. Institute of Technology, Australia.

Prove that \( L \) has the same last digit (i.e., units digit) for all \( n \) in the infinite geometric progression

\[ 4, 8, 16, 32, \ldots \]

B-383 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.

Solve the difference equation

\[ U_{n+2} - 5U_{n+1} + 6U_n = F_n \]

B-384 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.

Establish the identity

\[ p^n_{n+10} = 55(p^n_{n+8} - p^n_{n+2}) - 385(p^n_{n+6} - p^n_{n+4}) + p^n_0. \]

B-385 Proposed by Herta T. Freitag, Roanoke, VA.

Let \( T_n = n(n+1)/2 \). For how many positive integers \( n \) does one have both \( 10^6 < T_n < 2 \cdot 10^6 \) and \( T_n \equiv 3 \pmod{10} \)?

B-386 Proposed by Lawrence Somer, Washington, D.C.

Let \( p \) be a prime and let the least positive integer \( m \) with \( F_m \equiv 0 \pmod{p} \) be an even integer \( 2k \). Prove that \( F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p} \). Generalize to other sequences, if possible.

B-387 Proposed by George Berzsenyi, Lamar University, Beaumont, TX.

Prove that there are infinitely many ordered triples of positive integers \((x, y, z)\) such that

\[ 3x^2 - y^2 - z^2 = 1. \]
SOLUTIONS

ALMOST ALWAYS COMPOSITE

B-358 Proposed by Phil Mana, Albuquerque, New Mexico.

Prove that the integer $u_n$ such that $u_n \leq \frac{n^2}{3} < u_n + 1$ is a prime for only a finite number of positive integers $n$. (Note that $u_n = \lfloor \frac{n^2}{3} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer in $x$ and $u_1 = 0$, $u_2 = 1$, $u_3 = 3$, $u_4 = 5$, and $u_5 = 8$.)

Solution by Graham Lord, Université Laval, Québec.

If $n = 3m$, $3m + 1$, or $3m + 2$, where $m = 0, 1, 2, \ldots$, then $u_n = 3m^2$, $m(3m + 2)$ or $(m + 1)(3m + 1)$, respectively. Thus, the only values of $u_n$ that are prime are 3 and 5.

Also solved by George Berzsenyi, Paul S. Bruckman, Roger Engle & Sahib Singh, Herta T. Freitag, Bob Friellipp, and the proposer.

TRIBONACCI SEQUENCE

B-359 Proposed by R. S. Field, Santa Monica, CA.

Find the first three terms $T_1$, $T_2$, and $T_3$ of a Tribonacci sequence of positive integers $\{T_n\}$ for which

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{and} \quad \sum_{n=1}^{\infty} (T_n/10^n) = 1/T_1.$$

Solution by Graham Lord, Université Laval, Québec.

If $S(x) = \sum_{n=1}^{\infty} T_n x^n$, then

$$S(x) = [T_1(x - x^2 - x^3) + T_2(x^2 - x^3) + T_3 x]/(1 - x - x^2 - x^3),$$

and, in particular,

$$S(1/10) = (89T_1 + 9T_2 + T_3)/889.$$

Hence,

$$T_4(89T_1 + 9T_2 + T_3) = 889 = 7 \cdot 127.$$ 

Since $T_4 = T_3 + T_2 + T_1 \geq 3$, it must be the smaller prime factor, 7, and 

$$89T_1 + 9T_2 + T_3 = 127.$$ 

Thus, $T_1 = 1$, $T_2 = 4$, and $T_3 = 2$.

Also solved by George Berzsenyi, Michael Brozinski, Paul S. Bruckman, Roger Engle & Benjamin Freed & Sahib Singh, Charles B. Shields, and the proposer.

APPLYING QUATERNION NORMS

B-360 Proposed by T. O'Callahan, Aerojet Manufacturing Co., Fullerton, CA.

Show that for all integers $a, b, c, d, e, f, g, h$ there exist integers $\omega, x, y, z$ such that

$$(a^2 + 2b^2 + 3c^2 + 6d^2)(e^2 + 2f^2 + 3g^2 + 6h^2) = (\omega^2 + 2x^2 + 3y^2 + 6z^2).$$

Solution by Roger Engle & Sahib Singh, Clarion State College, Clarion, PA.
Defining the real quaternions $A$ and $B$ as
\[ A = a + (\sqrt{2}b)i + (\sqrt{3}c)j + (\sqrt{6}d)k, \]
\[ B = e + (\sqrt{2}f)i + (\sqrt{3}g)j + (\sqrt{6}h)k \]
and using the multiplicative property of norm $N$, namely $N(AB) = N(A)N(B)$, we conclude by comparison that
\[ w = ae - 2bf - 3ag - 6dh, \quad x = af + be + 3ch - 3dg, \]
\[ y = ag - 2bh + ce + 2df, \quad z = ah + bg - cf + de. \]

Also solved by Paul S. Bruckman, Bob Prielipp, Gregory Wulczyn, and the proposer.

A RATIONAL FUNCTION

B-361 Proposed by L. Carlitz, Duke University, Durham, N.C.

Show that
\[ \sum_{r,s=0}^{\infty} x^r y^s u^{\min(r,s)} v^{\max(r,s)} \]
is a rational function of $x$, $y$, $u$, and $v$ when these four variables are less than 1 in absolute value.

Solution by Roger Engle & Sahib Singh, Clarion State College, Clarion, PA.

If $S$ denotes the required sum, then
\[ S = \sum_{i=0}^{\infty} (xv)^i + \sum_{i=1}^{\infty} (yu)^i + xuvS \]
\[ \therefore S(1 - xuv) = \frac{1}{1 - xv} + \frac{yu}{1 - yv} \]
\[ \therefore S = \frac{1 - xuv^2}{(1 - xv)(1 - yv)(1 - xyuv)} \]

Also solved by Paul S. Bruckman, Robert M. Giuliani, Graham Lord, and proposer.

TRIANGULAR NUMBER RESIDUES

B-362 Proposed by Herta T. Freitag, Roanoke, VA.

Let $m$ be an integer greater than one (1) and let $R_n$ be the remainder when the triangular number $T_n = n(n+1)/2$ is divided by $m$. Show that the sequence $R_0$, $R_1$, $R_2$, ... repeats in a block $R_0$, $R_1$, ..., $R_{\ell}$ which reads the same from right to left as it does from left to right. (For example, if $m = 7$ then the smallest repeating block is 0, 1, 3, 6, 3, 1, 0.)

Solution by Graham Lord, Université Laval, Québec.

Since $T_{n+2m} = T_n + m(2n + 1 + 2m)$ then $R_n = R_{n+2m}$: the sequence repeats in blocks. And for $0 \leq n < m$, as $T_{2m-n-1} = T_n + m(2m - 2n - 1)$ it follows that $R_n = R_{2n-n-1}$, which implies the reflecting property.

Note that if $m$ is even the period is $2m$, since neither $T_n$ nor $T_{\ell}$ is congruent to 0 modulo $m$. And if $m$ is odd the period is $m$. The latter is proven
thus: As $T_{n+m} \equiv T_n \pmod{m}$, the period, $d$, must divide $m$. But, by the reflecting property and the periodicity $T_0 \equiv T_{d-1} \equiv T_d \pmod{m}$, that is, $m$ divides $T_d - T_{d-1} = d$. Hence, $d = m$.

Also solved by George Berzsenyi, Paul S. Bruckman, Roger Engle & Sahib Singh, Bob Prielipp, Gregory Wulczyn, and the proposer.

OVERLAPPING PALINDROMIC BLOCKS

B-363 Proposed by Herta T. Freitag, Roanoke, VA.

Do the sequences of squares $S_n = n^2$ and of pentagonal numbers $P_n = n(3n - 1)/2$ also have the symmetry property stated in B-362 for their residues modulo $m$?

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA.

For this symmetry property, it is necessary that two consecutive members of $S_n$ or $P_n$ be congruent to zero modulo $m$.

(a) $S_n = n^2, S_{n+1} = (n + 1)^2$.

Since $(n, n + 1) = 1$, $S_n$ does not have the symmetry property of B-362.

(b) $P_n = \frac{n}{2}(3n - 1), P_{n+1} = \frac{n + 1}{2}(3n + 2), P_n = 1, 5, 12, 22, 35, \ldots$.

For any factor $m$ of $n$, $(n, n + 1) = 1$, $(n, 3n + 2) = 1, 2$.
For any factor $m$ of $3n - 1$, $(3n - 1, 3n + 2) = 1, (3n - 1, n + 1) = 1, 2, 4$.

Since the only common factor to $P_n$ and $P_{n+1}$ is 2, $P_n$ does not have the symmetry property of B-362.

Also solved by Paul S. Bruckman, Roger Engle & Sahib Singh, Graham Lord, Bob Prielipp, and the proposer.

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