ON NTH POWERS IN THE LUCAS AND FIBONACCI SERIES

RAY STEINER

Bowling Green State University, Bowling Green, Ohio 43402

A. INTRODUCTION

Let F_n be the nth term in the Fibonacci series defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,$$

and let L_n be the nth term in the Lucas series defined by

$$L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$$

In a previous paper [3], H. London and the present author considered the problem of finding all the Nth powers in the Lucas and Fibonacci series. It was shown that the problem reduces to solving certain Diophantine equations, and all the cubes in both series were found. However, the problem of finding all the cubes in the Fibonacci sequence depended upon the solutions of the equations $y^2 \pm 100 = x^3$, and the finding of all these solutions is quite a difficult matter.

In the present paper we first present a more elementary proof of this fact which does not depend on the solution of $y^2 \pm 100 = x^3$. We then show that if p is a prime and $p \geq 5$, then L_{3k} and L_{2k} are never pth powers. Further, we show that if $F_{2^t k}$ is a pth power then $t \leq 1$, and we find all the 5th powers in the sequence F_{2m}. Finally, we close with some discussion of Lucas numbers of the form $y^p + 1$.

In our work we shall require the following theorems, which we state without proof:

Theorem 1: The Lucas and Fibonacci numbers satisfy the relations

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad \text{and} \quad L_m F_m = F_{2m}.$$

Theorem 2 (Nagell [6]): The equation $Ax^3 + By^3 = C$, where $3 \nmid AB$ if $C = 3$ has at most one solution in nonzero integers (u, v). There is a unique exception for the equation $x^3 + 2y^3 = 3$, which has exactly the two solutions $(u, v) = (1, 1)$ and $(-5, 4)$.

Theorem 3 (Nagell [7, p. 28]): If n is an odd integer ≥ 3, A is a square-free integer ≥ 1, and the class number of the field $Q(\sqrt{-A})$ is not divisible by n then the equation $Ax^2 + 1 = y^n$ has no solutions in integers x and y for y odd and ≥ 1 apart from $x = \pm 11$, $y = 3$ for $A = 2$ and $n = 5$.

Theorem 4 (Nagell [7, p. 29]): Let n be an odd integer ≥ 3 and let A be a square-free integer ≥ 3. If the class number of the field $Q(\sqrt{-A})$ is not divisible by n, the equation $Ax^2 + 4 = y^n$ has no solutions in odd integers A, x, and y.

Theorem 5 (Af Ekenstam [1], p. 5]): Let ε be the fundamental unit of the ring $R(\sqrt{m})$. If $N(\varepsilon) = -1$, the equation $x^{2n} - My^{2n} = 1$ has no integer solutions with $y \neq 0$.

Theorem 6 [5, p. 301]: Let p be an odd prime. Then the equation $y^2 + 1 = x^p$ is impossible for $x > 1$.

B. CUBES IN THE LUCAS SEQUENCE

By Theorem 1, we have

(1) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

If $L_n = y^3$, $F_n = x$, we get

$$y^6 - 5x^2 = 4(-1)^n$$

with $u > 0$ and $x > 0$. Suppose first n is even, then we get

$$y^6 - 4 = 5x^2.$$

If y is even, this equation is impossible mod 32. Thus, y is odd, and

$$(y^3 + 2)(y^3 - 2) = x^2$$

with

$$(y^3 + 2,\ y^3 - 2) = 1;$$

this implies that either

$$\begin{cases} y^3 + 2 = u^2 \\ y^3 - 2 = 5v^2 \end{cases}$$

or

$$\begin{cases} y^3 + 2 = 5u^2 \\ y^3 - 2 = v^2. \end{cases}$$

But it is well known (see, e.g., [9], pp. 399-400) that the only solution of the equation $y^3 + 2 = u^2$ is $y = -1$, $u = 1$. This, however, does not yield any value for v. Further, the equation $y^3 - 2 = v^2$ has only the solution $v = \pm 5$, $y = 3$. But this does not yield a value for u. Therefore, there are no cubes in the sequence L_{2m}.

Note: This result also follows immediately from Theorem 5 since the class number of $Q(\sqrt{5})$ is 1.

Next, suppose n is odd, then we get

(2) $$5x^2 - 4 = y^6.$$

If y is even, this equation is impossible mod 32. Thus x and y are odd, and (2) reduces to

(3) $$5x^2 - 4 = u^3,$$

with u a square. Equation (3) may be written

$$(2 + \sqrt{5}x)(2 - \sqrt{5x}) = u^3,$$

and since x is odd,

$$(2 + \sqrt{5}x,\ 2 - \sqrt{5x}) = 1.$$

Thus we conclude

(4) $$2 + \sqrt{5}x = \left(\frac{a + b\sqrt{5}}{2}\right)^3$$

or

(5) $$2 + \sqrt{5}x = \left(\frac{a + b\sqrt{5}}{2}\right)^3 \left(\frac{1 + \sqrt{5}}{2}\right).$$

Equation (4) yields

$$3a^2b + 5b^3 = 8,$$

which, in turn, yields

$$a = b = 1,\ v = 1.$$

To solve (5), we note that

$$N\left(\frac{a + b\sqrt{5}}{2}\right) = u, \text{ i.e., } a^2 - 5b^2 = 4u.$$

Since u is odd, a and b are odd, thus $\left(\frac{a + b\sqrt{5}}{2}\right)^3$ is of the form $S + T\sqrt{5}$, with S even and T odd. And

$$\left(\frac{1 + \sqrt{5}}{2}\right)(S + T\sqrt{5})$$

can never be of the form $A + B\sqrt{5}$. Thus, (5) is impossible. We have proved

Theorem 7: The only cube in the Lucas sequence is $L_1 = 1$.

C. CUBES IN THE FIBONACCI SEQUENCE

First, suppose m is even. Then $F_{2n} = x^3$ implies $F_nL_n = x^3$. If $n \not\equiv 0 \pmod 3$, $(F_n, L_n) = 1$. Thus, $L_n = t^3$ and $n = 1$.

If $n \equiv 0 \pmod 3$, $(F_n, L_n) = 2$ and either:

a. L_n is a cube, which is impossible;
b. $L_n = 2z^3$, $F\ = 4y^3$; or
c. $L_n = 4z^3$, $F\ = 2y^3$.

We now use equation (1) and first suppose n even. Then case (b) reduces to

$$z^3 - 20y^3 = 1,$$

with y and z squares. By Theorem 2, the only solutions of this equation are $z = 1$, $y = 0$, and $z = -19$, $y = -7$. Of these, only the first yields a sqaure for z and we get $n = 0$. Case (c) reduces to solving

$$4z^3 - 5y^3 = 1,$$

with y and z squares. Again, by Theorem 2, the only solution of this equation is $z = -1$, $y = -1$, which does not yield a square value for z. If n is odd, we get the two equations

$$z^3 - 20y^3 = -1 \quad\text{and}\quad 4z^3 - 5y^3 = -1$$

with z and y squares. By the results above, the only solutions of these two equations are $(z, y) = (-1, 0)$, $(19, 7)$, and $(1, 1)$. Of these, only the last yields a square value of z and we get $n = 3$. Thus, the only cubes in the sequence F_{2n} are $F_0 = 0$, $F_2 = 1$, and $F_6 = 8$.

If m is odd, and $F_m = x^3$, F_m cannot be even since then (1) yields

$$L_m^2 - 5x^6 = -4,$$

which is impossible (mod 32). Thus, the problem reduces to

$$10y^3 - 8 = 2x^2, \tag{6}$$

with x and y odd. But (6) was solved completely in [4, pp. 107-110]. We outline the solution here. Let $\theta^3 = 10$, where θ is real. We use the fact that $Z[\theta]$ is a unique factorization domain and apply ideal factorization theory to reduce this problem to

$$y\theta - 2 = (-2 + \theta)\left(\frac{a + b\theta + c\theta^2}{3}\right) \tag{7}$$

and three other equations, all of which may be proved impossible by congruence conditions.

To solve (7), we equate coefficients of θ and θ^2 to get

$$a^2 + 20bc - 5b^2 - 10ac = 9, \tag{8}$$

and

$$b^2 + 2ac = ab + 5c^2. \tag{9}$$

Equation (9) may be written

$$(b + 2c - a)(b - 2c) = c^2,$$

and (8) and (9) yield a odd, b even, and c even. Thus, we conclude

$$\begin{aligned} b - a + 2c &= dh_1^2 \\ b - 2c &= 4dh_2^2 \\ c &= 2dh_1h_2\varepsilon \end{aligned} \tag{10}$$

where $\varepsilon = \pm 1$, d, h_1, h_2 are rational integers, and $d > 0$. If we solve (10) for a, b, c and substitute in (8) we get

$$h^4 + 4h_1^3h_2\varepsilon - 24h_1^2h_2^2 - 16h_1h_2\varepsilon - 64h_2^4 = 9/d^2,$$

which reduces to

$$u^4 - 30u^2v^2 + 40uv^3 - 75v^4 = 9/d^2. \tag{11}$$

If $d = 1$, (11) is impossible (mod 5). If $d = 3$, (11) may be written

$$(u - 5v)(u^3 + 5u^2v - 5uv^2 + 15v^3) = 1,$$

and from this it follows easily that the only integer solutions of (11) are $(u, v) = (\pm 1, 0)$. Thus, we have:

Theorem 8: The only cubes in the Fibonacci sequence are

$$F_0 = 0,\ F_1 = F_2 = 1, \text{ and } F_6 = 8.$$

D. HIGHER POWERS IN THE LUCAS AND FIBONACCI SEQUENCES

In this section, we investigate the problem of finding all pth powers in the Fibonacci and Lucas sequences, where p is a prime and $p \geq 5$. We show that L_{3n} and L_{2n} are never pth powers, and that if F_{2^tn} is a pth power, then $t = 0$ or 1. We conclude by finding all the 5th powers in the sequence F_{2n}.

Suppose $L_n = x^p$; then equation (1) yields

$$x^{2p} - 5F_n^2 = 4(-1)^n.$$

If $n \equiv 0 \pmod 3$, x and F_{2n} are even, and this equation is impossible (mod 32) regardless of the parity of n.

Suppose further that $n = 2m$, $m \not\equiv 0 \pmod 3$; then (1) yields

$$5F_m^2 + 4 = x^{2p},$$

with x and F_m odd. Since the class number of $Q(\sqrt{-5})$ is 2, this equation has no solutions with F_m odd. Thus, we have:

Theorem 9: L_{3k} and L_{2k} are never pth powers for any k. Finally, if n is odd, $n \not\equiv 0 \pmod 3$, we have to solve

$$5y^2 - 4 = x^{2p}.$$

Unfortunately, known methods of treating this equation lead to the solution of irreducible equations. Thus, it is quite difficult to solve.

Now let us suppose $F_n = x^p$. If $n \equiv 3 \pmod 6$, F_n is even and Equation (1) becomes

$$L_n^2 - 5x^{2p} = -4,$$

which is impossible (mod 32). Thus, F_{6k+3} is never a pth power for any k.

Now we prove:

Theorem 10: If F_{6k} is a pth power, then $k = 0$.

Proof: Let $m = 3k$, then $(F_m, L_m) = 2$, and since L_{3k} is not a pth power, we conclude:

a. $$\begin{cases} L_m = 2u^p \\ F_m = 2^{rp-1}v^p, \end{cases}$$

or

b. $$\begin{cases} L_m = 2^{rp-1}u^p \\ F_m = 2v^p, \end{cases}$$

with u and v odd, m even, and $r \geq 1$.

Now note that m cannot be odd in either case, since then (1) yields

$$L_m^2 - 5F_m^2 = -4 = L_m^2 - 5 \cdot 2^t v^{2p} \text{ for some integer } t,$$

which is impossible (mod 32).

If we substitute $F_m = 2v^p$ in (1), case (b) reduces to

$$x^2 - 1 = 5v^{2p}, \tag{12}$$

since m is even. If v is odd, (12) is impossible (mod 8). Thus, x is odd, v is even, and (12) yields

$$\begin{cases} \dfrac{x+1}{2} = u^{2p} \\ \dfrac{x-1}{2} = 5v^{2p}, \end{cases}$$

i.e.,

$$u^{2p} - 5v^{2p} = 1.$$

Since the fundamental unit of $Z[1, \sqrt{5}]$ is $2 + \ 5$ and $N(2 + \sqrt{5}) = -1$, this equation has no solution for $v \neq 0$, by Theorem 5. The solution $v = 0$ yields $L_0 = 2$ and $p = 2$, which is impossible. Thus, $F_{6k} \neq 2v^p$ for any $k \neq 0$.

To solve case (a), suppose $m \neq 0$ and $m = 2^t \ell$, ℓ odd, $\ell \equiv 0 \pmod 3$. Then

$$F_{2^t \ell} = F_{2^{t-1}\ell} L_{2^{t-1}\ell} = 2^{rp-1}v^p.$$

Since $F_{2^{t-1}\ell} \neq 2v^p$ and $L_{2^{t-1}\ell}$ is not a pth power, we conclude

$$L_{2^{t-1}\ell} = 2u_1^p,$$
$$F_{2^{t-1}\ell} = 2^{rp-1}v_1^p,$$

with u_1 and v_1 odd. By continuing this process, we eventually get

$$F_{2^s \ell} = y^p \quad \text{for some } s \text{ and } y \text{ odd},$$
$$F_\ell = y^p \quad \text{for } y \text{ odd, or}$$
$$F_\ell = 2^s y^p \quad \text{for some } s \text{ and } y \text{ odd}.$$

Since $F_{2^s \ell}$ and F_ℓ are even if $\ell \equiv 0 \pmod 3$, the first two of these equations are impossible.

To settle the third, note that we have

$$L_\ell = 2x^p$$
$$F_\ell = 2^s y^p$$

with ℓ odd, $\ell \equiv 0 \pmod 3$ and x and y odd. Then (1) yields

$$x^{2p} - 5 \cdot 4^{s-1} y^{2p} = -1.$$

If $s > 1$, this equation is impossible mod 20; if $s = 1$, we get

$$x^{2p} - 5y^{2p} = -1,$$

which is impossible mod 4, since x and y are odd. Thus, $m = 0$.

Next, suppose $m \not\equiv 0 \pmod 3$, and F_{2m} is a pth power. Then

$$F_{2m} = F_m L_m, \quad (F_m, L_m) = 1$$

and F_m and L_m are both pth powers. This enables us to prove the following result:

Theorem 11: If $F_{2^t m}$ is a pth power, m odd, $m \not\equiv 0 \pmod 3$, then $t \leq 1$.

Proof: Suppose $F_{2^t m}$ is a pth power with m odd, $m \not\equiv 0 \pmod 3$ and $t > 1$. Then

$$F_{2^t m} = F_{2^{t-1} m} L_{2^{t-1} m}$$

and both $L_{2^{t-1} m}$ and $F_{2^{t-1} m}$ are pth powers. Further,

$$F_{2^{t-1} m} = F_{2^{t-2} m} L_{2^{t-2} m},$$

and both $F_{2^{t-2} m}$ and $L_{2^{t-2} m}$ are powers. By continuing this process, we eventually get

$$F_{4m} = F_{2m} L_{2m},$$

and both F_{2m} and L_{2m} are pth powers. This is impossible by Theorem 9, and thus $t \leq 1$.

If m is odd, $m \not\equiv 0 \pmod 3$ and F_{2m} is a pth power, then

$$F_{2m} = F_m L_m$$

and both F_m and L_m are pth powers. Thus, we must solve

$$x^{2p} - 5y^{2p} = -4. \tag{13}$$

Unfortunately, it seems quite difficult to solve this equation for arbitrary p. We shall give the solution for $p = 5$ presently, and shall return to (13) in a future paper.

Finally, if m is odd and $F_m = x^p$, we have to solve

$$x^{2p} + 4 = 5y^{2p},$$

with y odd. Again, the solution of this equation leads to irreducible equations and is thus quite difficult to solve. To conclude this section, we prove

Theorem 12: The only 5th power in the sequence F_{2m} is $F_2 = 1$.

Proof: if we substitute $p = 5$ in (13), it reduces to

$$x^5 + 5y^5 = 4, \tag{14}$$

and we must prove that the only integer solution of this equation is $x = -1$,

$y = 1$. To this end, we consider the field $Q(\theta)$, where $\theta^5 = 5$. We find that an integral basis is $(1, \theta, \theta^2, \theta^3, \theta^4)$ and that a pair of fundamental units is given by

$$\varepsilon_1 = 1 - 10 - 5\theta^2 + 3\theta^3 + 4\theta^4,$$

$$\varepsilon_2 = -24 + 15\theta - 5\theta^2 - 2\theta^3 + 5\theta^4.$$

Since $(2, -1 + \theta)^2 = (-1 + \theta)$, the only ideal of norm 4 is $(-1 + \theta)$. Thus, (14) reduces to

$$u + v\theta = (-1 + \theta)\varepsilon_1^m \varepsilon_2^n. \tag{15}$$

We now use Skolem's method [8]. We find

$$\varepsilon_1^5 = 1 + 5\xi_1$$

with

$$\xi_1 = 3\theta^3 + 4\theta^4 + 5A$$

and

$$\varepsilon_2^5 = 1 + 5\xi_2$$

with

$$\xi_2 = 3\theta^3 + 5B,$$

where A and B are elements of $Z[\theta]$.

If we write $m = 5u + r$, $n = 5v + s$ and treat (15) as a congruence (mod 5), we find that it holds only for $r = s = 0$. Thus, (15) may be written

$$\begin{aligned} u + v\theta &= (-1 + \theta)(1 + 5\xi_1)^u (1 + 5\xi_2)^v \\ &= (-1 + \theta)[1 + 5(u\xi_1 + v\xi_2) + \cdots]. \end{aligned}$$

Now we equate the coefficients of θ^3 and θ^4 to 0 and get

$$-3u + 3v + 0(5) = 0,$$

$$-u + 3v + 0(5) = 0.$$

Since $$\begin{vmatrix} -3 & 3 \\ -1 & 3 \end{vmatrix} \not\equiv 0 \pmod 5,$$

the equation

$$u + v\theta + w\theta^2 = (-1 + \theta)\varepsilon_1^m \varepsilon_2^n$$

has no solution except $m = n = 0$, when $m \equiv n \equiv 0 \pmod 5$, by a result of Skolem [8] and Avanesov [2]. Thus, the only solution of (15) is $m = n = 0$, and the result follows.

E. LUCAS NUMBERS OF THE FORM $y^p + 1$

For our final result, we prove

Theorem 13: Let p be an odd prime. If $L_{2m} = y^p + 1$, then $m = 0$.

Proof: Again, we use (1). We set $L_{2m} = y^p + 1$, $F_{2m} = x$, and get

$$(y^p + 1)^2 - 4 = 5x^2,$$

i.e.,

$$y^{2p} + 2y^p - 3 = 5x^2,$$

i.e.,

$$(y^p + 3)(y^p - 1) = 5x^2.$$

The GDC of $y^p + 3$ and $y^p - 1$ divides 4. If y is even, both these numbers are relatively prime, and we get

$$\begin{cases} y^p + 3 = u^2, \\ y^p - 1 = 5v^2, \end{cases}$$

or

$$\begin{cases} y^p + 3 = 5u^2, \\ y^p - 1 = v^2. \end{cases}$$

Since y is even, both these systems are impossible (mod 8).

Suppose next that $y \equiv 3 \pmod 4$. Then $(y^p + 3,\ y^p - 1) = 2$, and we get

$$\begin{cases} y^p + 3 = 2u^2, \\ y^p - 1 = 10v^2, \end{cases}$$

or

$$\begin{cases} y^p + 3 = 10u^2, \\ y^p - 1 = 2v^2. \end{cases}$$

By Theorem 3, the equation $y \quad - 1 = 10v^2$ has no solution with y odd for $y \neq 1$ since the class number of $Q(\sqrt{-10})$ is 2. But $y = 1$ contradicts $y = 3 \pmod 4$. Further, $y^p - 1 = 2v$ has no solution with y odd except $y = 1$, $v = 0$, and $y = 3$, $v = \pm 11$, $p = 5$. But $y = 3$ does not yield a value for u.

Finally, if $y \equiv 1 \pmod 4$, $(y^p + 3,\ y^p - 1) = 4$, and we get

$$\begin{cases} y^p + 3 = 20u^2, \\ y^p - 1 = 4v^2, \end{cases}$$

or

$$\begin{cases} y^p + 3 = 4u^2, \\ y^p - 1 = 20v^2. \end{cases}$$

By Theorem 6, $y^p - 1 = 4v^2$ has no integer solution except $y = 1$, $v = 0$. However, this does not yield a value for u. By Theorem 3, the only solution of $y^p - 1 = 20v^2$ with y odd is $y = 1$, $v = 0$, and we get

$$y = 1,\ u = 1, \text{ and } x = 0.$$

The result follows.

REFERENCES

1. A. Af Ekenstam, "Contributions to the Theory of the Diophantine Equation $Ax^n - By^n = C$" (Inaugural Dissertation, Uppsala, 1959).
2. E. T. Avanesov, "On a Question of a Certain Theorem of Skolem," *Akad. Nauk. Armjan. SSR* (Russian), Ser. Mat. 3, No. 2 (1968), pp. 160-165.
3. H. London & R. Finkelstein, "On Fibonacci and Lucas Numbers Which Are Perfect Powers," *The Fibonacci Quarterly*, Vol. 7, No. 4 (1969), pp. 476-481.
4. H. London & R. Finkelstein, *On Mordell's Equation* $y^2 - k = x^3$ (Bowling Green, Ohio: Bowling Green University Press, 1973).
5. L. J. Mordell, *Diophantine Equations* (New York: Academic Press, 1969).
6. T. Nagell, "Solution complète de quelques équations cubiques à deux indéterminées," *J. Math. Pure Appl.*, Vol. 9, No. 4 (1925), pp. 209-270.
7. T. Nagell, "Contributions to the Theory of a Category of Diophantine Equations of the Second Degree with Two Unknowns," *Nova Acta Reg. Soc. Sci. Uppsala*, Vol. 4, No. 2 (1955), p. 16.

8. T. Skolem, "Ein Verfahren zur Behandlung gewisser exponentialler Gleichungen und diophantischer Gleichungen," *8de Skand. Mat. Kongress Stockholm* (1934), pp. 163-188.
9. J. V. Uspensky & M. A. Heaslet, *Elementary Number Theory* (New York: McGraw-Hill Book Company, 1939).

*****

GENERALIZED TWO-PILE FIBONACCI NIM

JIM FLANIGAN
University of California at Los Angeles, Los Angeles, CA 90024

1. INTRODUCTION

Consider a take-away game with one pile of chips. Two players alternately remove a positive number of chips from the pile. A player may remove from 1 to $f(t)$ chips on his move, t being the number removed by his opponent on the previous move. The last player able to move wins.

In 1963, Whinihan [3] revealed winning strategies for the case when $f(t) = 2t$, the so-called *Fibonacci Nim*. In 1970, Schwenk [2] solved all games for f nondecreasing and $f(t) \geq t\ \forall t$. In 1977, Epp & Ferguson [1] extended the solution to the class where f is nondecreasing and $f(1) \geq 1$.

Recently, Ferguson solved a *two-pile analogue of Fibonacci Nim*. This motivated the author to investigate take-away games with more than one pile of chips. In this paper, winning strategies are presented for a class of two-pile take-away games which *generalize* two-pile Fibonacci Nim.

2. THE TWO-PILE GAME

Play begins with two piles containing m and m' chips and a positive integer w. Player I selects a pile and removes from 1 to w chips. Suppose t chips are taken. Player II responds by taking from 1 to $f(t)$ chips from one of the piles. We assume *f is nondecreasing* and $f(t) \geq t\ \forall t$. The two players alternate moves in this fashion. The player who leaves both piles empty is the winner. If $m = m'$, Player II is assured a win.

Set $d = m' - m$. For $d \geq 1$, define $L(m, d)$ to be the least value of w for which Player I can win. Set $L(m, 0) = \infty\ \forall m \geq 0$. One can systematically generate a *tableau* of values for $L(m, d)$. Given the position (m, d, w), the player about to move can win iff he can:

(1) take t chips, $1 \leq t \leq w$, from the large pile, leaving the next player in position $(m, d - t, f(t))$ with $f(t) < L(m, d - t)$; or

(2) take t chips, $1 \leq t \leq w$, from the small pile, leaving the next player in position $(m - t, d + t, f(t))$ with $f(t) < L(m - t, d + t)$.

(See Fig. 2.1.) Consequently, the tableau is governed by the functional equation

$$L(m, d) = \min\left\{t > 0 \,\middle|\, f(t) < L(m, d - t) \text{ or } f(t) < L(m - t, d + t)\right\}$$

subject to $L(m, 0) = +\infty\ \forall m \geq 0$. Note that $L(m, d) \leq d\ \forall d \geq 1$. Dr. Ferguson has written a computer program which can quickly furnish the players with a 60×40 tableau. As an illustration, Figure 2.2 gives a tableau for the two-pile game with $f(t) = 2t$, two-pile Fibonacci Nim.