ON KTH-POWER NUMERICAL CENTERS

RAY STEINER
Bowling Green State University, Bowling Green, Ohio 43402

1. INTRODUCTION

In a previous paper [2], we considered the problem of determining all positive integers which possess kth-power numerical centers, and proved that there are infinitely many positive integers possessing first-power numerical centers and that the only positive integer possessing a second-power numerical center is 1. In the present paper, we treat the cases $k = 3$, 4, and 5.

2. THE CASE $k = 3$

Let us begin by recalling the following

Definition: Given the positive integer n, we call the positive integer N, ($N \leq n$), a kth-power numerical center for n in case the sum of the kth powers of the integers from 1 to N equals the sum of the kth powers from N to n.

In this section, we prove the following

Theorem 1: The only positive integer possessing a third-power numerical center is 1.

Proof: Let N be any third-power numerical center for the positive integer n. Since the sum of the cubes of the first N positive integers is given by

$$\sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4},$$

the condition that N be a third-power numerical center for n requires that

$$\frac{N^2(N+1)^2}{4} = \frac{n^2(n+1)^2}{4} - \frac{N^2(N-1)^2}{4}.$$

On setting $X = 2N^2 + 1$, we obtain

$$X^2 - 2n^2(n+1)^2 = 1. \tag{1}$$

Let us now consider the following

Problem: To find all triangular numbers whose square is also triangular. This requires

$$\left(\frac{n(n+1)}{2}\right)^2 = \frac{N(N+1)}{2}. \tag{2}$$

and, on setting $X = 2n + 1$, we again obtain equation (1). But equation (2) was solved by Ljunggren [3] and Cassels [1], who showed that its only positive integer solutions are $(n, N) = (1, 1)$ and $(3, 8)$. Thus, the only positive integer solutions of (1) with X odd are $(x, n) = (3, 1)$ and $(17, 3)$.

From this, it follows that the only positive integer solution of (1) satisfying $X = 2n^2 + 1$ is $(X, n) = (3, 1)$, and our result is proved.

3. THE CASES $k = 4$ AND 5

Since

$$\sum_{i=1}^{N} i^4 = \frac{N(N+1)(6N^3 + 9N^2 + N - 1)}{30},$$

the condition that N be a fourth-power numerical center for n requires

$$N(N+1)(6N^3 + 9N^2 + N - 1)$$
$$= n(n+1)(6n^3 + 9n^2 + n - 1) - N(N-1)(6N^3 - 9N^2 + N + 1),$$

and on setting $X = 2n + 1$, $Y = 2N$, we obtain

(3) $$3X^5 - 10X^3 + 7X = 6Y^5 + 40Y^3 - 16Y$$

subject to the conditions

(4) X positive and odd, Y positive and even.

Further, since

$$\sum_{i=1}^{N} i^5 = \frac{N^2(N^2 + 2N + 1)(2N^2 + 2N - 1)}{12}$$

the condition that N be a fifth-power numerical center for n requires

$$N^2(N^2 + 2N + 1)(2N^2 + 2N - 1)$$
$$= n^2(n^2 + 2n + 1)(2n^2 + 2n - 1) - N^2(N^2 - 2N + 1)(2N^2 - 2N - 1),$$

and, on setting

$$X = (2n + 1)^2,\ Y = (2N)^2$$

it reduces to

(5) $$X^3 - 5X^2 + 7X - 3 = 2Y^3 + 20Y^2 - 16Y$$

subject to the conditions

(6) X a positive odd square, Y a positive even square.

Unfortunately, we have been unable to discover a method of solving equations (3) and (5) completely, although we have used a computer to verify that the only integer solution of (3), subject to (4), with $X < 205$ is $(X, Y) = (3, 2)$ and that the only integer solution of (5), subject to (6), with $X < 411$ is $(X, Y) = (9, 4)$.

REFERENCES

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