We show that these ranks are bounded both above and below. Since $x_1 \le a$, the number of rows a partition of n can occupy is not less than u, where

$$u - 1 < n/a \le u.$$

Hence, none of the ranks can exceed $(a - u)$.

Similarly, none of the ranks can fall short of $(v - b)$, where

$$v - 1 < n/b \le v.$$

Of course, for n to have a partition of said type, it is necessary to have

$$n \le ab.$$

REFERENCE

1. A. O. L. Atkin, "A Note on Ranks and Conjugacy of Partitions," *Quart. J. Math.*, Vol. 17, No. 2 (1966), pp. 335-338.

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THE ANDREWS FORMULA FOR FIBONACCI NUMBERS

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1. In what follows: small letters denote integers; $n > 0$; p denotes an odd prime other than 5; [ ] is the greatest integer function; and for convenience, we write

$$(n;r) \text{ for } \binom{n}{r}.$$

The two relations

$$(n;r) = (n;n - r), \text{ and} \tag{1.1}$$

$$(n;r - 1) + (n;r) = (n + 1;r) \tag{1.2}$$

are freely used, and we take, as usual,

$(t;0) = 1$ for all integers t, and

$(n;r) = 0$ if $r > n$, and also when r is negative.

We further define

$$S(n,r) = \sum_j (n;j), \tag{1.3}$$

where j runs over all nonnegative integers which are $\equiv r \pmod 5$.

As a consequence of this definition and the relations (1.1) and (1.2) we have

$$S(n,r) = S(n,n - r), \text{ and} \tag{1.4}$$

$$S(n,r - 1) + S(n,r) = S(n + 1,r). \tag{1.5}$$

2. The Fibonacci numbers F_n are defined by the relations

$$F_1 = 1 = F_2, \text{ and} \tag{2.1}$$

$$F_n + F_{n+1} = F_{n+2} \text{ for each } n \ge 1. \tag{2.2}$$

G. E. Andrews [1] has given the following formulas for F_n:

$$F_n = \sum_j (-1)^j \big(n - 1; [(n - 1 - 5j)/2]\big); \tag{2.3}$$

$$F_n = \sum_j (-1)^j \big(n; [(n - 1 - 5j)/2]\big); \tag{2.4}$$

where j runs over the set of integers.

The object of this note is to provide a simple proof of these formulas and to obtain some congruence properties of F_n. Let

$$[(n - 1)/2] \equiv m \pmod 5$$

so that

$$n - 1 = 2m \quad \text{or} \quad 2m + 1 \pmod{10}$$

according as n is odd or even. Then (2.3) and (2.4) can be written as:

$$F_n = S(n - 1, m) - S(n - 1, m - 2); \tag{2.5}$$

$$F_n = S(n, m) - S(n, m - 1). \tag{2.6}$$

We first assert that (2.5) and (2.6) are equivalent and prove the assertion as follows:

For any integer j, we have

$$(n; m + 5j) - (n - 1; m + 5j) = (n - 1; m + 5j - 1).$$

Also

$$\begin{aligned}(n; m - 1 + 5j) &- (n - 1; m - 2 + 5j)\\ &= (n; n - m + 1 - 5j) - (n - 1; n - m + 1 - 5j)\\ &= (n - 1; n - m - 5j)\\ &= (n - 1; m + 5j - 1).\end{aligned}$$

Hence, letting j vary suitably, we get

$$S(n, m) - S(n - 1, m) = S(n, m - 1) - S(n - 1, m - 2),$$

and our assertion follows immediately.

3. Proof of (2.5) is by induction. It is easy to verify that (2.5) and (2.6) hold for $n = 1$ and $n = 2$. Assume that they hold for each $n \leq t + 1$. Then, from (2.6), we have

$$F_t = S(t, m) - S(t, m - 1) \tag{3.1}$$

with $m \equiv [(t - 1)/2] \pmod 5$. For the same m, (2.5) gives

$$F_{t+1} = S(t, m) - S(t, m - 2) \text{ for } t \text{ odd,} \tag{3.2}$$

$$\qquad = S(t, m + 1) - S(t, m - 1) \text{ for } t \text{ even.} \tag{3.3}$$

If t is odd, let $t = 10k + 2m + 1$; then

$$S(t, m) = S(t, t - m) = S(t, 10k + m + 1) = S(t, m + 1). \tag{3.4}$$

If t is even, let $t = 10k + 2m + 2$; then

$$S(t, m - 1) = S(t, t - m + 1) = S(t, 10k + m + 3) = S(t, m - 2); \tag{3.5}$$

so that

$$F_{t+1} = S(t, m + 1) - S(t, m - 2) \text{ for } t \text{ odd as well as } t \text{ even.} \tag{3.6}$$

From (3.1) and (3.6), we get

$$F_t + F_{t+1} = \{S(t,m) + S(t,m+1)\} - \{S(t,m-1) + S(t,m-2)\}$$
$$= S(t+1,m+1) - S(t+1,m-1).$$

Thus,

$$F_{t+2} = S(t+1,m+1) - S(t+1,m-1).$$

Inductive reasoning now proves (2.5) for all $n > 0$.

4. From (2.5) and (2.6), we can derive not only the well-known congruences modulo p for F_p, F_{p+1}, and F_{p-1} (in the manner of Andrews), but also some congruences modulo p^2.
We first give the expressions for F_{p^2}, F_{p^2+1}, and F_{p^2-1}.

(i) If p is a prime of the form $10k \pm 1$, then we have

$$[(p^2-1)/2] \equiv 0 \pmod{5},$$

and so also

$$[p^2/2] \equiv 0 \pmod{5}.$$

Hence,

$$F_{p^2} = S(p^2, 0) - S(p^2, 4),$$
$$F_{p^2+1} = S(p^2, 0) - S(p^2, 3);$$

and therefore,

$$F_{p^2-1} = S(p^2, 4) - S(p^2, 3).$$

(ii) If p is a prime of the form $10k \pm 3$, then

$$[(p^2-1)/2] \equiv 4 \pmod{5},$$

and so also is

$$[p^2/2] \equiv 4 \pmod{5}.$$

Hence,

$$F_{p^2} = S(p^2, 4) - S(p^2, 3),$$
$$F_{p^2+1} = S(p^2, 4) - S(p^2, 2);$$

and therefore,

$$F_{p^2-1} = S(p^2, 3) - S(p^2, 2).$$

All that we need now for our purpose is the

Lemma: For $1 \leq h \leq p^2 - 1$,

$$(p^2; h) \equiv (-1)^{h-1}p^2/h \pmod{p^2}.$$

Proof: We have

$$(p^2; h) = \frac{p^2}{h} \cdot \frac{p^2-1}{1} \cdot \frac{p^2-2}{2} \cdot \ldots \cdot \frac{p^2-h+1}{h-1}.$$

Since for $1 \leq r \leq h-1$,

$$\frac{p^2-r}{r} \equiv -1 \pmod{p^2}$$

the lemma follows immediately.

Evidently, if $p \nmid h$, then

(4.1) $$(p^2;h) \equiv 0 \pmod{p^2};$$

otherwise,

(4.2) $$(p^2;h)/p \equiv (-1)^{h-1}p/h \pmod{p}.$$

We have, of course,

(4.3) $$(p^2;0) = 1 = (p^2;p^2).$$

As an application of the lemma, we have, for example:

(i) when $1 \le m \le 4$,

(4.4) $$S(p^2,m) \equiv \sum_{j \ge 0} (p^2;m + 5j) \pmod{p^2}.$$

On the right of the sigma in (4.4), we need consider only those nonnegative values of j for which

$$m + 5j \le p^2 \text{ and } m + 5j \equiv 0 \pmod{p};$$

(ii) when $m = 0$, we have,

(4.5) $$S(p^2,0) - 1 \equiv \sum_{j \ge 1} (p^2;5j) \pmod{p^2},$$

so that

(4.6) $$\frac{S(p^2,0) - 1}{p} \equiv \sum_{j} (-1)^{j-1}/5j \pmod{p},$$

where $1 \le j < p/5$. Thus

$$\frac{F_{121} - 1}{11} \equiv \frac{1}{5} - \frac{1}{10} + \frac{1}{4} - \frac{1}{9} \equiv 9 - 10 + 3 - 5 \equiv 8 \pmod{11}.$$

Therefore,

$$F_{121} \equiv 89 \pmod{121}.$$

REFERENCE

1. G. E. Andrews, *The Fibonacci Quarterly*, Vol. 7, No. 2 (1969), pp. 113-130.

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OPERATIONAL FORMULAS FOR UNUSUAL FIBONACCI SERIES

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Operational formulas can play a fascinating role in finding transformations and sums of series. For instance, by using the differential operator D ($=d/dx$) we can transform

(1) $$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}, \quad |x| < 1,$$

into

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}, \quad |x| < 1.$$