So,

$$\begin{aligned} y(n+1)+1 &= 3x(n+1) - y(n) + 1 \\ &= 3\big(x(n+1)+1\big) - \big(y(n)+1\big) - 1 \\ &= 3f(2n+3)f(2n+4) - f(2n+2)f(2n+3) - 1 \\ &= 2f(2n+3)f(2n+4) + f(2n+3)\big(f(2n+2)+f(2n+3)\big) \\ &\qquad - f(2n+2)f(2n+3) - 1 \\ &= 2f(2n+3)f(2n+4) + \big(f^2(2n+3) - 1\big) \\ &= 2f(2n+3)f(2n+4) + f(2n+2)f(2n+4) \\ &= f(2n+3)f(2n+4) + f^2(2n+4) \\ &= f(2n+4)f(2n+5), \end{aligned}$$

completing the proof.

REFERENCES

1. W. H. Mills, "A Method for Solving Certain Diophantine Equations," *Proc. Amer. Math. Soc.* 5 (1954):473-475.
2. James C. Owings, Jr., "An Elementary Approach to Diophantine Equations of the Second Degree," *Duke Math. J.* 37 (1970):261-273.

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THE DIOPHANTINE EQUATION $Nb^2 = c^2 + N + 1$

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Other than $b = c = 0$ (in which case $N = -1$), the Diophantine equation $Nb^2 = c^2 + N + 1$ has no solutions. This family of equations includes the 1976 Mathematical Olympiad problem $a^2 + b^2 + c^2 = a^2b^2$ (letting $N = a^2 - 1$) and such problems as $6b^2 = c^2 + 7$, $a^2b^2 = a^2 + c^2 + 1$, etc.

Noting that $b^2 \neq 1$ (since $N \neq c^2 + N + 1$), one may restate the problem as follows:

$$\begin{aligned} Nb^2 &= c^2 + N + 1 \\ Nb^2 - N &= c^2 + 1 \\ N(b^2 - 1) &= c^2 + 1 \\ N &= (c^2 + 1)/(b^2 - 1). \end{aligned}$$

Thus the problem reduces to showing that, except as noted, $(c^2 + 1)/(b^2 - 1)$ cannot be an integer. [This result domonstrates the interesting fact that $c^2 \not\equiv -1 \pmod{b^2 - 1}$, i.e., that none of the Diophantine equations $c^2 \equiv 2 \pmod 3$, $c^2 \equiv 7 \pmod 8$, etc., has a solution.]

It is well known [1, p. 25] that for any prime p, $p \mid c^2 + 1 \Rightarrow p = 2$ or $p = 4m + 1$.*

$$\begin{aligned} b^2 - 1 \mid c^2 + 1 \Rightarrow b^2 - 1 &= 2^s(4m_1 + 1)(4m_2 + 1) \cdots (4m + 1) \\ &= 2^s(4M + 1) \\ b^2 &= 2^s(4M) + 2^s + 1 \end{aligned}$$

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*The result of this article is not merely a special case of this theorem [e.g., according to the theorem $(c^2 + 1)/8$ could be an integer].

$$s \neq 0, \text{ since } s = 0 \Rightarrow b^2 = 4M + 2$$
$$\Rightarrow b^2 \text{ is even}$$
$$\Rightarrow b \text{ is even}$$
$$(b/2)(b) \text{ is even}$$
$$\text{but } (b/2)(b) = b^2/2 = 2M + 1, \text{ which is odd}$$
$$s > 0 \Rightarrow b^2 \text{ is odd}$$
$$\Rightarrow b \text{ is odd, so let } b = 2k + 1$$

$$(2k + 1)^2 = 2^s(4M) + 2^s + 1$$
$$4k^2 + 4k + 1 = 2^s(4M) + 2^s + 1$$
$$4(k^2 + k - 2^s M) = 2^s$$
$$\Rightarrow s \geq 2$$
$$\Rightarrow 4 \text{ is a factor of } b^2 - 1$$
$$\Rightarrow 4 \mid c^2 + 1$$
$$\Rightarrow c^2 + 1 = 4n$$
$$c^2 = 4n - 1$$
$$\Rightarrow c^2 \text{ is odd}$$
$$\Rightarrow c \text{ is odd, so let } c = 2h + 1$$
$$(2h + 1)^2 = 4n - 1$$
$$4h^2 + 4h + 1 = 4n - 1$$
$$4h^2 + 4h + 2 = 4n$$
$$2h^2 + 2h + 1 = 2n$$

But this is a contradiction (since the right-hand side of the equation is even, and the left-hand side of the equation is odd). So, $(c^2 + 1)/(b^2 - 1)$ cannot be an integer, and the Diophantine equation $Nb^2 = c^2 + N + 1$ has no nontrivial solution.

Following through the above proof, one can readily generalize

$$Nb^2 = c^2 + N + 1$$

to

$$Nb^2 = c^2 + N(4k + 1) + 1.$$

Just letting $N = 1$, one includes in the above result such Diophantine equations as

$$b^2 - c^2 = 6, \quad b^2 - c^2 = 10,$$

and, in general,

$$b^2 - c^2 \equiv 2 \pmod 4.$$

REFERENCE

1. I. M. Niven & H. S. Zuckerman, *Theory of Numbers* (New York: Wiley, 1960).

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