ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-394 *Proposed by Phil Mana, Albuquerque, NM.*

Let $P(x) = x(x-1)(x-2)/6$. Simplify the following expression:

$P(x+y+z) - P(y+z) - P(x+z) - P(x+y) + P(x) + P(y) + P(z)$.

B-395 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Let $c = (\sqrt{5}-1)/2$. For $n = 1, 2, 3, \ldots$, prove that

$$1/F_{n+2} < c^n < 1/F_{n+1}.$$

B-396 *Based on the solution to B-371 by Paul S. Bruckman, Concord, CA.*

Let $G_n = F_n(F_n+1)(F_n+2)(F_n+3)/24$. Prove that 60 is the smallest positive integer m such that $10|G_n$ implies $10|G_{n+m}$.

B-397 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Find a closed form for the sum

$$\sum_{k=0}^{2s}\binom{2s}{k}F_{n+kt}^2.$$

B-398 *Proposed by Herta T. Freitag, Roanoke, Va.*

Is there an integer K such that

$$K - F_{n+6} + \sum_{j=1}^{n} j^2F_j$$

is an integral multiple of n for all positive integers n?

B-399 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Let $f(x) = u_1 + u_2x + u_3x^2 + \cdots$ and $g(x) = v_1 + v_2x + v_3x^2 + \cdots$, where $u_1 = u_2 = 1$, $u_3 = 2$, $u_{n+3} = u_{n+2} + u_{n+1} + u_n$, and $v_{n+3} = v_{n+2} + v_{n+1} + v_n$. Find initial values v_1, v_2, and v_3 so that $e^{g(x)} = f(x)$.

SOLUTIONS

Nonhomogeneous Difference Equation

B-370 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Solve the difference equation: $u_{n+2} - 5u_{n+1} + 6u_n = F_n$.

Solution by Phil Mana, Albuquerque, NM.

Let E be the operator with $Ey_n = y_{n+1}$. The given equation can be rewritten as

$$(E - 2)(E - 3)U_n = F_n.$$

Operating on both sides of this with $(E - a)(E - b)$, where a and b are the roots of $x^2 - x - 1 = 0$, one sees that the solutions of the original equation are among the solutions of

$$(E - a)(E - b)(E - 2)(E - 3)U_n = 0.$$

Hence, $U_n = ha^n + kb^n + 2^n c + 3^n d$. Here, c and d are arbitrary constants. But h and k can be determined using $n = 0$ and $n = 1$, and one finds that $ha^n + kb^n = L_{n+3}/5$. Thus, $U_n = (L_{n+3}/5) + 2^n c + 3^n d$.

Also solved by Paul S. Bruckman, C. B. A. Peck, Bob Prielipp, Sahib Singh, and the proposer.

No, No, Not Always

B-371 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let $S_n = \sum_{k=1}^{F_n} \sum_{j=1}^{k} T_j$, where T_j is the triangular number $j(j + 1)/2$. Does each of $n \equiv 5 \pmod{15}$ and $n \equiv 10 \pmod{15}$ imply that $S_n \equiv 0 \pmod{10}$? Explain.

I. Solution by Sahib Singh, Clarion College, PA.

The answer to both questions is in the negative as explained below:

$$\sum_{j=1}^{k} T_j = \sum_{j=1}^{k} \binom{j+1}{2} = \binom{k+2}{3}$$

$$S_n = \sum_{k=1}^{F_n} \binom{k+2}{3} = \binom{F_n + 3}{4} = F_n(F_n + 1)(F_n + 2)(F_n + 3)/24.$$

One can show that $S_{25} \not\equiv 0 \pmod{10}$ and $S_{35} \not\equiv 0 \pmod{10}$ even though $25 \equiv 10 \pmod{15}$ and $35 \equiv 5 \pmod{15}$.

II. From the solution by Paul S. Bruckman, Concord, CA.

It can be shown that $S \equiv 0 \pmod{10}$ if and only if $n \equiv r \pmod{60}$ where $r \in \{0, 5, 6, 7, 10, 12, 17, 18, 20, 24, 29, 30, 31, 32, 34, 36, 43, 44, 46, 53, 54, 56, 58\}$.

Also solved by Bob Prielipp, Gregory Wulcyzn, and the proposer.

Still No

B-372 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let S_n be as in B-371. Does $S_n \equiv 0 \pmod{10}$ imply that n is congruent to either 5 or 10 modulo 15? Explain.

Solution by Paul S. Bruckman, Concord, CA.

$S_6 = \binom{F_6 + 3}{4} = \binom{11}{4} = 11 \cdot 10 \cdot 9 \cdot 8/24 = 330 \equiv 0 \pmod{10}$ but 6 is not congruent to 5 or 10 modulo 15.

Also solved by Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

Golden Cosine

B-373 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA and P. L. Mana, Albuquerque, NM.*

The sequence of Chebyshev polynomials is defined by

$$C_0(x) = 1,\ C_1(x) = x,\ \text{and}\ C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x)$$

for $n = 2, 3, \ldots$. Show that $\cos[\pi/(2n + 1)]$ is a root of

$$[C_{n+1}(x) + C_n(x)]/(x + 1) = 0$$

and use a particular case to show that $2\cos(\pi/5)$ is a root of

$$x^2 - x - 1 = 0.$$

Solution by A. G. Shannon, Linacre College, University of Oxford.

It is known that if $x = \cos\theta$ then $C_n(x) = \cos n\theta$. Letting

$$\theta = \pi/(2n + 1),$$

one has

$$x + 1 = \cos\theta + 1 \neq 0$$

and

$$C_{n+1}(x) + C_n(x) = \cos[(n + 1)\pi/(2n + 1)] + \cos[n\pi/(2n + 1)]$$
$$= -\cos[n\pi/(2n + 1)] + \cos[n\pi/(2n + 1)] = 0$$

as required, since $\cos(\pi - \alpha) = -\cos\alpha$.

The special case $n = 2$ shows us that $\cos(\pi/5)$ is a solution of

$$[C_3(x) + C_2(x)]/(x + 1) = 0,$$

which turns out to be

$$(2x)^2 - 2x - 1 = 0.$$

Hence, $2\cos(\pi/5)$ satisfies $x^2 - x - 1 = 0$.

Also solved by Paul S. Bruckman, Bob Prielipp, Sahib Singh, and the proposer.

Fibonacci in Trigonometric Form

B-374 *Proposed by Frederick Stern, San Jose State University, San Jose, CA.*

Show both of the following:

$$F_n = \frac{2^{n+2}}{5}\left[\left(\cos\frac{\pi}{5}\right)^n \sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \left(\cos\frac{3\pi}{5}\right)^n \sin\frac{3\pi}{5}\sin\frac{9\pi}{5}\right],$$

$$F_n = \frac{(-2)^{n+2}}{5}\left[\left(\cos\frac{2\pi}{5}\right)^n \sin\frac{2\pi}{5}\sin\frac{6\pi}{5} + \left(\cos\frac{4\pi}{5}\right)^n \sin\frac{4\pi}{5}\sin\frac{12\pi}{5}\right].$$

Solution by A. G. Shannon, Linacre College, University of Oxford.

Let $x_n = [2\cos(\pi/5)]^n$ and $y_n = [2\cos(3\pi/5)]^n$. It follows from B-373 that $x_{n+2} = x_{n+1} + x_n$, and it follows similarly that $y_{n+2} = y_{n+1} + y_n$. Hence the first result in this problem is established by verifying it for $n = 0$ and $n = 1$ and then using the recursion formulas for F_n, x_n, and y_n. The second result follows from the first using

$$\cos(3\pi/5) = -\cos(2\pi/5) \quad\text{and}\quad \cos(\pi/5) = -\cos(4\pi/5).$$

Also solved by Sahib Singh, Herta T. Freitag, Bob Prielipp, Douglas A. Fults, Paul S. Bruckman, and the proposer.

Fibonacci or Nil

B-375 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Express $\dfrac{2^{n+1}}{5}\displaystyle\sum_{k=1}^{4}\left[\left(\cos\frac{k\pi}{5}\right)\cdot\sin\frac{k\pi}{5}\cdot\sin\frac{3k\pi}{5}\right]$ in terms of Fibonacci number F_n.

Solution by Herta T. Freitag, Roanoke, VA.

Using the relationships established in B-374, the expression of this problem becomes $F_n[1 + (-1)\ ]/2$, which is F_n for even n and zero for odd n.

Also solved by Paul S. Bruckman, Douglas A. Fults, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

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