If the initial position and velocity of the jth mass are, respectively, X_j and V_j, then the normal coordinates are [6, p. 431]

$$\begin{aligned}\zeta_k(t) &= Re\sum_{j=1}^{N} ma_{jk}\, e^{i\omega_k t}\left(X_j - \frac{i}{\omega_k}V_j\right) \\ &= Re\sum_{j=1}^{N} m(-1)^{k-1} a_{j1} U_k\left(\cos\frac{2k\pi}{2N+1}\right)\exp\left[2i\omega_0 t\cos\frac{k\pi}{2N+1}\right] \\ &\qquad \times\left(X_j - \frac{iV_j}{2\omega_0\cos\frac{k\pi}{2N+1}}\right)\end{aligned} \tag{17}$$

REFERENCES

1. M. Bicknell, *The Fibonacci Quarterly* 8, No. 5 (1970):407.
2. V. E. Hoggatt, Jr., & D. A. Lind, *The Fibonacci Quarterly* 5, No. 2 (1967): 141.
3. U. W. Hochstrasser, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (U.S. Department of Commerce, National Bureau of Standards, Washington, D.C., 1964), p. 787.
4. M. Gardner, *Scientific American* 201 (1959):128.
5. B. Davis, *The Fibonacci Quarterly* 10, No. 7 (1972):659.
6. J. Marion, *Classical Dynamics of Particles and Systems* (2nd ed.; New York: Academic Press, 1970), p. 425.

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CONGRUENCES FOR CERTAIN FIBONACCI NUMBERS

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The purpose of this note is to prove some of the well-known congruences for the Fibonacci numbers U_p and U_{p-1}, where p is prime and $p \equiv \pm 1 \pmod 5$. We also prove a congruence which is analogous to

$$U_n = \frac{\alpha^\mu - \beta^\mu}{\alpha - \beta},\text{ where } \alpha \text{ and } \beta \text{ are the roots of } x^2 - x - 1 = 0.$$

We start by considering the congruence

(1) $x^2 - x - 1 \equiv 0 \pmod p$, which can also be written

(2) $y^2 \equiv 5 \pmod p$,

on putting $2x - 1 = y$.

It is well known that 5 is a quadratic residue of primes of the form $5m \pm 1$ and a quadratic nonresidue of primes of the form $5m \pm 3$. Therefore, (2) has a solution p if p is a prime and $p \equiv \pm 1 \pmod 5$.

It also has $-y$ as a solution, and these solutions are different in the sense that

$$y \not\equiv -y \pmod p.$$

This obviously gives two different solutions x_1 and x_2 of (1).

(1) is now written

(3) $$x^2 \equiv x + 1 \pmod{p},$$

or, which is the same,

$$X^2 \equiv U_1 X + U_2 \pmod{p},$$

where U_1 and U_2 are the first and second Fibonacci numbers.

When multiplied by x, (3) gives

$$x^3 \equiv x^2 + x \equiv x + 1 + x \equiv 2x + 1 \pmod{p},$$

or, which is the same,

$$X^3 \equiv U_3 X + U_2 \pmod{p}.$$

Suppose, therefore, that

(4) $$X_k \equiv U_k X + U_{k-1} \pmod{p} \text{ for some } k.$$

Now (4) implies

$$X^{k+1} \equiv U_k X^2 + U_{k-1} X \equiv U_k (X + 1) + U_{k-1} X \equiv (U_{k-1} + U_k) X + U_k$$

$$= U_{k+1} X + U_k \pmod{p},$$

which, together with (3) shows that (4) holds for $k \geq 2$.

For the two solutions x_1 and x_2, we now have

$$X_1^k \equiv U_k X_1 + U_{k-1} \pmod{p}$$

and

$$X_2^k \equiv U_k X_2 + U_{k-1} \pmod{p}.$$

Subtraction gives

(5) $$X_1^k - X_2^k \equiv U_k (X_1 - X_2) \pmod{p}.$$

Putting $k = p - 1$ in (5) and using Fermat's theorem, we get

$$X_1^{p-1} - X_2^{p-1} \equiv U_{p-1}(X_1 - X_2) \equiv 1 - 1 = 0 \pmod{p}.$$

Since $X_1 \not\equiv X_2 \pmod{p}$, this proves

$$U_{p-1} \equiv 0 \pmod{p}.$$

Putting $k = p$ in (5), we get in the same manner

(6) $$X_1^p - X_2^p \equiv X_1 - X_2 \equiv U_p (X_1 - X_2) \pmod{p},$$

which proves

$$U_p \equiv 1 \pmod{p}.$$

At last, (6) can formally be written

$$U_p \equiv \frac{X_1^p - X_2^p}{X_1 - X_2} \pmod{p},$$

which shows the analogy with the formula

$$U_n = \frac{\alpha^\mu - \beta^\mu}{\alpha - \beta}.$$

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