we may deduce that, if $P(N)$ denotes the period (mod N) of the Fibonacci and Lucas sequence (the periods for the two sequences *are* the same, except when $5|N$, cf. [2]), and if p is any odd prime $\neq 5$, then

(34) $$p(p^n) \text{ divides } \frac{1}{2}\left(3p + 1 - (p + 3)\left(\frac{5}{p}\right)\right)p^{n-1},\quad n = 1, 2, 3, \ldots .$$

We will leave the proof of this result to the reader.

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A NOTE ON A PELL-TYPE SEQUENCE

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The Pell sequence is defined by the recursive relation

$$P_1 = 1,\ P_2 = 2, \text{ and } P_{n+2} = 2P_{n+1} + P_n, \text{ for } n \geq 1.$$

The first few terms of the sequence are 1, 2, 5, 12, 29, 70, 169, 408, ... It is well known that the nth term of the Pell sequence can be written

$$P_n = \frac{1}{\sqrt{8}}\left[\left(\frac{2+\sqrt{8}}{2}\right)^n - \left(\frac{2-\sqrt{8}}{2}\right)^n\right].$$

It is also easily proven that $\lim_{n\to\infty} \frac{P_n}{P_{n+1}} = \frac{-2+\sqrt{8}}{2}$.

For the sequence $\{V_n\}$ defined by the recursive formula

$$V_1 = 1,\ V_2 = 2, \text{ and } V_{n+2} = kV_{n+1} + V_n, \text{ for } k \geq 1,$$

we know that

$$\lim_{n\to\infty} \frac{V_n}{V_{n+1}} = \frac{-k+\sqrt{k^2+4}}{2}.$$

If we let $k = 1$, the sequence $\{V_n\}$ becomes the Fibonacci sequence and the limit of the ratio of consecutive terms is $\frac{-1+\sqrt{5}}{2} = .618$, which is the "golden ratio." For $k = 2$ the ratio becomes .4142, which is the limit of the ratio of consecutive terms of the Pell sequence.

Both of the previous sequences were developed by adding two terms of a sequence or multiples of two terms to generate the next term. We now consider the ratio of consecutive terms of the sequence $\{G_n\}$ defined by the recursive formula

$$G_1 = a_1,\ G_2 = a_2,\ \ldots,\ G_n = a_n, \text{ and}$$

and

$$G_{n+1} = na_n + (n - 1)a_{n-1} + (n - 2)a_{n-2} + \cdots + 2a_2 + a_1$$

where a_i is an integer > 0.

Suppose that when this sequence is continued a sufficient number of terms it is possible to find n consecutive terms such that the limit of the ratio of any two consecutive terms approaches r. The sequence could be written

$$G_m, \frac{G_m}{r}, \frac{G_m}{r^2}, \frac{G_m}{r^3}, \ldots, \frac{G_m}{r^{n-1}}.$$

The next term, $\frac{G_m}{r^n}$, may be written as

$$\frac{G_m}{r^n} = n\left(\frac{G_m}{r^{n-1}}\right) + (n - 1)\left(\frac{G_m}{r^{n-2}}\right) + \cdots + 2\frac{G_m}{r} + G_m.$$

Simplifying,

$$G_m = nrG_m + (n - 1)r^2G_m + \cdots + 2r^{n-1}G_m + r^nG_m.$$

Dividing by G_m, we obtain

$$1 = nr + (n - 1)r^2 + \cdots + 2r^{n-1} + r^n$$

or

(1) $$r^n + 2r^{n-1} + \cdots + (n - 2)r^3 + (n - 1)r^2 + nr - 1 = 0.$$

The limiting value of r is seen to be the root of equation 1.

If we let $n = 4$, $G_1 = 2$, $G_2 = 4$, $G_3 = 3$, and $G_4 = 1$, the corresponding sequence is 2, 4, 3, 1, 23, 105, 494, 2338, 11067, 52375, ... The ratios of consecutive terms are

$$\frac{2}{4} = 0.5000 \qquad \frac{105}{494} = 0.2125$$

$$\frac{4}{3} = 1.3333 \qquad \frac{494}{2338} = 0.2113$$

$$\frac{3}{1} = 3.0000 \qquad \frac{2338}{11067} = 0.2113$$

$$\frac{1}{23} = 0.0434 \qquad \frac{11067}{52375} = 0.2113$$

$$\frac{23}{105} = 0.2190$$

The computed ratio approaches .2113. Using equation 1 we have, for this sequence, $r^4 + 2r^3 + 3r^2 + 4r - 1 = 0$. By successive approximation, we find $r \approx .2113$. The reader may also wish to verify this conclusion for other initial values for the sequence as well as for a different number of initial terms.

REFERENCE

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