since

$$\frac{F_n}{L_{n-1}} = \frac{F_n}{F_{n-1}}\,\frac{F_{n-1}}{L_{n-1}} = L^2.$$

4. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN A MULTINOMIAL TRIANGLE

We challenge the reader to find the generating functions of the $(H - L)/k$ sequences in the quadrinomial triangle. We surmise that the limits would be the generating functions of the central values in Pascal's quadrinomial triangle.

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SOLUTION OF $\binom{y+1}{x} = \binom{y}{x+1}$ IN TERMS OF FIBONACCI NUMBERS

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In [2, pp. 262-263] we solved the Diophantine equation $\binom{y+1}{x} = \binom{y}{x+1}$ and found that (x,y) is a solution iff for some $n \geq 0$,

$$(x + 1, y + 1) = \left(\sum_{k=0}^{n} f(4k + 1),\ \sum_{k=0}^{n} f(4k + 3)\right),$$

where $$f(0) = 0,\ f(1) = 1,\ f(n + 2) = f(n) + f(n + 1).$$

We show here that (x,y) is a solution iff for some $n \geq 0$,

$$(x + 1, y + 1) = \big(f(2n + 1)f(2n + 2),\ f(2n + 2)f(2n + 3)\big),$$

incidentally deriving the identities

$$f(2n + 1)f(2n + 2) = \sum_{k=0}^{n} f(4k + 1),$$

$$f(2n + 2)f(2n + 3) = \sum_{k=0}^{n} f(4k + 3).$$

Briefly, in [2], we solved $\binom{y+1}{x} = \binom{y}{x+1}$ as follows. When multiplied out this equation becomes

$$x^2 + y^2 - 3xy - 2x - 1 = 0.$$

Now, if (x,y) is a solution of this polynomial equation, so are (x',y) and (x,y'), where $x' = -x + 3y + 2$ and $y' = -y + 3x$, because

$$\begin{aligned} 0 &= x^2 + y^2 - 3xy - 2x - 1 = y^2 + x(x - 3y - 2) - 1 \\ &= y^2 + x(-x') - 1 = y^2 + x'(-x) - 1 \\ &= y^2 + x'(x' - 3y - 2) - 1 = (x')^2 + y^2 - 3x'y - 2x' - 1, \end{aligned}$$

and similarly for (x,y'). So from the basic solution $x = 0$, $y = 1$ we get the four-tuple

$$(y',x,y,x') = (-1,0,1,5)$$

in which each adjacent pair of integers forms a solution. Repeating the process gives

$$(-1,-1,0,1,5,14);$$

doing it twice more we get

$$(-3,-2,-1,-1,0,1,5,14,39,103).$$

We have now found three solutions to $\binom{y+1}{x} = \binom{y}{x+1}$, namely (0,1), (5,14), (39,103). In [2] we showed, with little trouble, that all integral solutions to the given polynomial equation may be found somewhere in the two-way infinite chain generated by (0,1). (See Mills [1] for the genesis of this type of argument.) Hence (x,y) is a solution to the binomial equation iff $0 \le x < y$ and (x,y) occurs somewhere in this chain. If we let

$$\big(x(0),y(0)\big) = (0,1),\ \big(x(1),y(1)\big) = (5,14),\ \text{etc.},$$

and use our equations for x' and y', we find that

$$x(n + 1) = -x(n) + 3y(n) + 2,$$

$$y(n + 1) = -y(n) + 3x(n).$$

(WARNING: In [2] the roles of x and y are reversed.)

We prove our assertion by induction on n, appealing to the well-known identities

$$f^2(2n + 2) + 1 = f(2n + 1)f(2n + 3),$$

$$f^2(2n + 1) - 1 = f(2n)f(2n + 2).$$

Obviously, $x(0) + 1 = f(1)f(2)$, $y(0) + 1 = f(2)f(3)$. So assume

$$\big(x(n) + 1, y(n) + 1\big) = \big(f(2n + 1)f(2n + 2), f(2n + 2)f(2n + 3)\big).$$

Then

$$\begin{aligned} x(n + 1) + 1 &= 3y(n) - x(n) + 3 = 3\big(y(n + 1) + 1\big) - \big(x(n) + 1\big) + 1 \\ &= 3f(2n + 2)f(2n + 3) - f(2n + 1)f(2n + 2) + 1 \\ &= 2f(2n + 2)f(2n + 3) + f(2n + 2)\big(f(2n + 1) + f(2n + 2)\big) \\ &\qquad - f(2n + 1)f(2n + 2) + 1 \\ &= 2f(2n + 2)f(2n + 3) + \big(f^2(2n + 2) + 1\big) \\ &= 2f(2n + 2)f(2n + 3) + f(2n + 1)f(2n + 3) \\ &= f(2n + 2)f(2n + 3) + f^2(2n + 3) = f(2n + 3)f(2n + 4). \end{aligned}$$

So,

$$\begin{aligned} y(n+1) + 1 &= 3x(n+1) - y(n) + 1 \\ &= 3\big(x(n+1) + 1\big) - \big(y(n) + 1\big) - 1 \\ &= 3f(2n+3)f(2n+4) - f(2n+2)f(2n+3) - 1 \\ &= 2f(2n+3)f(2n+4) + f(2n+3)\big(f(2n+2) + f(2n+3)\big) \\ &\qquad - f(2n+2)f(2n+3) - 1 \\ &= 2f(2n+3)f(2n+4) + \big(f^2(2n+3) - 1\big) \\ &= 2f(2n+3)f(2n+4) + f(2n+2)f(2n+4) \\ &= f(2n+3)f(2n+4) + f^2(2n+4) \\ &= f(2n+4)f(2n+5), \end{aligned}$$

completing the proof.

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THE DIOPHANTINE EQUATION $Nb^2 = c^2 + N + 1$

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Other than $b = c = 0$ (in which case $N = -1$), the Diophantine equation $Nb^2 = c^2 + N + 1$ has no solutions. This family of equations includes the 1976 Mathematical Olympiad problem $a^2 + b^2 + c^2 = a^2b^2$ (letting $N = a^2 - 1$) and such problems as $6b^2 = c^2 + 7$, $a^2b^2 = a^2 + c^2 + 1$, etc.

Noting that $b^2 \neq 1$ (since $N \neq c^2 + N + 1$), one may restate the problem as follows:

$$\begin{aligned} Nb^2 &= c^2 + N + 1 \\ Nb^2 - N &= c^2 + 1 \\ N(b^2 - 1) &= c^2 + 1 \\ N &= (c^2 + 1)/(b^2 - 1). \end{aligned}$$

Thus the problem reduces to showing that, except as noted, $(c^2 + 1)/(b^2 - 1)$ cannot be an integer. [This result domonstrates the interesting fact that $c^2 \not\equiv -1 \pmod{b^2 - 1}$, i.e., that none of the Diophantine equations $c^2 \equiv 2 \pmod 3$, $c^2 \equiv 7 \pmod 8$, etc., has a solution.]

It is well known [1, p. 25] that for any prime p, $p \mid c^2 + 1 \Rightarrow p = 2$ or $p = 4m + 1$.*

$$\begin{aligned} b^2 - 1 \mid c^2 + 1 \Rightarrow b^2 - 1 &= 2^s(4m_1 + 1)(4m_2 + 1) \cdots (4m \;\; + 1) \\ &= 2^s(4M + 1) \\ b^2 &= 2^s(4M) + 2^s + 1 \end{aligned}$$

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*The result of this article is not merely a special case of this theorem [e.g., according to the theorem $(c^2 + 1)/8$ could be an integer].