ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-299 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Vandermonde determinants:

(A) Evaluate

$$\Delta = \begin{vmatrix} F_{2r} & F_{6r} & F_{10r} & F_{14r} & F_{18r} \\ F_{4r} & F_{12r} & F_{20r} & F_{28r} & F_{36r} \\ F_{6r} & F_{18r} & F_{30r} & F_{42r} & F_{54r} \\ F_{8r} & F_{28r} & F_{40r} & F_{56r} & F_{72r} \\ F_{10r} & F_{30r} & F_{50r} & F_{70r} & F_{90r} \end{vmatrix}$$

(B) Evaluate

$$D = \begin{vmatrix} 1 & L_{2r+1} & L_{4r+2} & L_{6r+3} & L_{8r+4} \\ 1 & -L_{6r+3} & L_{12r+6} & L_{18r+9} & L_{24r+12} \\ 1 & L_{10r+5} & L_{20r+10} & L_{30r+15} & L_{40r+20} \\ 1 & -L_{14r+7} & L_{28r+14} & -L_{42r+21} & L_{56r+28} \\ 1 & L_{18r+9} & L_{36r+18} & L_{54r+27} & L_{72r+36} \end{vmatrix}$$

(C) Evaluate

$$D_1 = \begin{vmatrix} 1 & L_{2r} & L_{4r} & L_{6r} & L_{8r} \\ 1 & L_{6r} & L_{12r} & L_{18r} & L_{24r} \\ 1 & L_{10r} & L_{20r} & L_{30r} & L_{40r} \\ 1 & L_{18r} & L_{36r} & L_{54r} & L_{72r} \end{vmatrix}$$

H-300 *Proposed by James L. Murphy, California State College, San Bernardino, CA*

Given two positive integers A and B relatively prime, form a "multiplicative" Fibonacci sequence $\{A_i\}$ with $A_1 = A$, $A_2 = B$, and $A_{i+2} = A^*A_{i+1}$. Now form the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{i=1}^{n} A_i.$$

$\{S_n\}$ is a subsequence of the arithmetic sequence $\{T_n\}$ where

$$T_n = A + nB,$$

and by Dirichlet's theorem we know that infinitely many of the T_n are prime. The question is: Does such a sparse subsequence $\{S_n\}$ of the arithmetic sequence $A + nB$ also contain infinitely many primes?

Notes:

$S_1 = A,\quad S_2 = A + B,\quad S_3 = A + B + AB,$

$S_4 = A + B + AB + AB^2,\quad S_5 = A + B + AB + AB^2 + A^2B^3,$ etc.

Some examples:

For $A = 2$ and $B = 3$, the first few S_i are:

2, 5, 11, 29, 137, 2081, all prime, and

$S_7 = 212033 = 43*4931.$

For $A = 3$ and $B = 14$, the first few S_i are:

3, 17, 59, 647, 25343, 14546591, all prime, and

$S_7 = 358631287199 = 43*8340262493.$

For $A = 2$ and $B = 21$, the first few S are:

2, 23, prime; $S_3 = 65$, a composite; but

$S_4 = 947$ and $S_5 = 37881$, both prime.

Looking at the first six terms of the sequence $\{S_i\}$ for 68 different choices of A and B, I found the following distribution:

Number of Primes in the First Six Terms	Number of Sequences Having This Number of Primes
1	2
2	19
3	21
4	22
5	2
6	2
	68

H-301 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $A_0, A_1, A_2, \ldots, A_n, \ldots$ be a sequence such that the nth differences are zero (that is, the Diagonal Sequence terminates). Show that, if

$$A(x) = \sum_{i=0}^{\infty} A_i x^i,$$

then

$$A(x) = \frac{1}{1-x} \cdot D\left(\frac{x}{1-x}\right), \qquad \text{where} \qquad D_n(x) = \sum_{i=0}^{\infty} d_i x^i.$$

SOLUTIONS

Pell Mell

H-275 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let P_n denote the Pell Sequence defined as follows:

$$P_1 = 1,\ P_2 = 2,\ P_{n+2} = 2P_{n+1} + P_n \quad (n \geq 1).$$

Consider the array below:

$$\begin{array}{ccccccccccc}
1 & & 2 & & 5 & & 12 & & 29 & & 70 \ldots (P_n) \\
& 1 & & 3 & & 7 & & 17 & & 41 \ldots & \\
& & 2 & & 4 & & 10 & & 24 \ldots & & \\
& & & 2 & & 6 & & 14 \ldots & & & \\
& & & & 4 & & 8 \ldots & & & & \\
& & & & & 4 & & & & &
\end{array}$$

Each row is obtained by taking differences in the row above.

Let D_n denote the left diagonal sequence in this array; i.s.,

$$D_1 = D_2 = 1,\ D_3 = D_4 = 2,\ D_5 = D_6 = 4,\ D_7 = D_8 = 8,\ \ldots .$$

(i) Show $D_{2n-1} = D_{2n} = 2^{n-1}$ $(n \geq 1)$.

(ii) Show that if $F(x)$ represents the generating function for $\left\{P_n\right\}_{n=1}^{\infty}$ and $D(x)$ represents the generating function for $\left\{D_n\right\}_{n=1}^{\infty}$, then

$$D(x) = F\left(\frac{x}{1+x}\right).$$

Solution by George Berzsenyi, Lamar University, Beaumont, TX

First observe that each row in the array inherits the recursive relation of the Pell numbers. This is true more generally, for if $\{x_n\}$ is a sequence defined recursively by

$$x_{n+2} = \alpha x_{n+1} + \beta x_n$$

and if $\{y_n\}$ is defined by

$$y_n = x_{n+1} - x_n,$$

then

$$\begin{aligned} y_{n+2} &= x_{n+3} - x_{n+2} = \alpha(x_{n+2} - x_{n+1}) + \beta(x_{n+1} - x_n) \\ &= \alpha y_{n+1} + \beta y_n. \end{aligned}$$

Let E_n be the second diagonal sequence in the array; i.e.,

$$E_1 = 2,\ E_2 = 3,\ E_3 = 4,\ E_4 = 6,\ E_5 = 8,\ \ldots .$$

We shall prove by induction that for each $n = 1, 2, \ldots, D_{2n-1} = D_{2n} = 2^{n-1}$, while $E_{2n-1} = 2 \cdot 2^{n-1}$ and $E_{2n} = 3 \cdot 2^{n-1}$. The portion of the array shown exhibits this fact for $n = 1$; assume it for $n = k$. Then the first few members of the $2k - 1$st and $2k$th rows can be obtained by using the recursion formula and upon taking differences one obtains the first two members of the next two rows as follows:

$$\begin{array}{ccccccccc}
2^{k-1} & & 2\cdot 2^{k-1} & & 5\cdot 2^{k-1} & & 12\cdot 2^{k-1} & & 29\cdot 2^{k-1} \\
& 2^{k-1} & & 3\cdot 2^{k-1} & & 7\cdot 2^{k-1} & & 17\cdot 2^{k-1} & \\
& & 2^k & & 2\cdot 2^k & & 5\cdot 2^k & & \\
& & & 2^k & & 3\cdot 2^k & & &
\end{array}$$

This completes the induction and establishes part (i).

To prove part (ii), recall that

$$F(x) = \frac{x}{1 - 2x - x^2},$$

and therefore,

$$F\left(\frac{x}{1 + x}\right) = \frac{x + x^2}{1 - 2x^2}.$$

On the other hand, if

$$D(x) = \sum_{n=1}^{\infty} D_n x^n,$$

then

$$D(x) = (x + x^2) + 2(x^3 + x^4) + 2^2(x^5 + x^6) + \cdots$$

while

$$-2x^2 D(x) = -2(x^3 + x^4) - 2^2(x^5 + x^6) - \cdots .$$

Hence, $(1 - 2x^2)D(x) = x + x^2$, and

$$D(x) = \frac{x + x^2}{1 - 2x^2}.$$

Consequently the desired relationship, $D(x) = F\left(\frac{x}{1 + x}\right)$ follows.

Also solved by V. E. Hoggatt, Jr., P. Bruckman, G. Wulczyn, and A. Shannon.

Late Acknowledgment: P. Bruckman solved H-274.

HELP WANTED

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AND

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