CONCAVITY PROPERTY AND A RECURRENCE RELATION FOR ASSOCIATED LAH NUMBERS

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ABSTRACT

A recurrence relation is obtained for the associated Lah numbers,

$L_k(m,n)$,

via their generating function. Using this result, it is shown that $L_k(m,n)$ is a strong logarithmic concave function of n for fixed k and m.

1. INTRODUCTION

The Lah numbers $L(m,n)$ (see Riordan [4, p. 44]) with arguments m and n are given by the relation

$$L(m,n) = (-1)^n (m!/n!)\binom{m-1}{n-1}, \tag{1}$$

where $L(m,n) = 0$ for $n > m$. Since the sign of $L(m,n)$ is the same as that of $(-1)^n$, we may write (1) in absolute value as

$$|L(m,n)| = (m!/n!)\binom{m-1}{n-1}. \tag{2}$$

We define the associated Lah numbers $L_k(m,n)$ for integral $k > 0$ as

$$L_k(m,n) = (m!/n!)\sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}\binom{m+rk-1}{m} \tag{3}$$

where $L_k(m,n) = 0$ for $n > m$. Using the binomial coefficient identity (12.13) in Feller [2, p. 64], it can be easily seen that

$$L_1(m,n) = |L(m,n)|. \tag{4}$$

The use of the associated Lah numbers $L_k(m,n)$ has recently arisen in a paper by the author [1], where the n-fold convolution of independent random variables having the decapitated negative binomial distribution is derived in terms of the numbers $L_k(m,n)$. In this paper, we first provide a recurrence relation for the numbers $L_k(m,n)$. This result is then utilized to show that $L_k(m,n)$ is a strong logarithmic concave (SLC) function of n for fixed k and m, that is, $L_k(m,n)$ satisfies the inequality

$$[L_k(m,n)]^2 > L_k(m,n+1)L_k(m,n-1) \tag{5}$$

for $k = 1, 2, \ldots, m = 3, 4, \ldots,$ and $n = 2, 3, \ldots, m-1$.

2. RECURRENCE RELATION FOR $L_k(m,n)$

The author [1] has provided a generating function for the numbers $L_k(m,n)$ in the form

$$[(1-\theta)^{-k} - 1]^n = \sum_{m=n}^{\infty} n! L_k(m,n)\theta^m/m!. \tag{6}$$

Differentiating both sides of (6) with respect to θ, then multiplying both sides by $(1 - \theta)$, gives

$$nk[(1-\theta)^{-k} - 1]^{n-1}(1-\theta)^{-k} = (1-\theta)\Sigma n!L_k(m,n)\theta^{m-1}/(m-1)! \tag{7}$$

which, using (6), becomes

$$\begin{aligned} nk\Sigma n!L_k(m,n)\theta^m/m! &+ nk\Sigma(n-1)!L_k(m,n-1)\theta^m/m! \\ &= (1-\theta)\Sigma n!L_k(m,n)\theta^{m-1}/(m-1)!. \end{aligned} \tag{8}$$

Now, equating the coefficient of θ^m in (8), we obtain the recurrence formula for $L_k(m,n)$ as

$$L_k(m+1,n) = (nk+m)L_k(m,n) + kL_k(m,n-1). \tag{9}$$

The recurrence relation (9) is used to obtain Table I for the associated Lah numbers $L_k(m,n)$ for $n = 1(1)5$ and $m = 1(1)5$. It may be remarked that, for $k = 1$, Table I reduces to the one for the absolute Lah numbers given in Riordan [4, p. 44].

3. CONCAVITY OF $L_k(m,n)$

The proof of the SLC property of the numbers $L_k(m,n)$ is based on the following result of Newton's inequality given in Hardy, Littlewood, and Polya [3, p. 52]: If the polynomial

$$P(x) = \sum_{n=1}^{m} c_n x^n$$

has only real roots, then

$$c_n^2 > c_{n+1}c_{n-1} \tag{10}$$

for $n = 2, 3, \ldots, m - 1$. To establish the SLC property, we need the following:

Lemma: If

$$P_m(x) = \sum_{n=1}^{m} L_k(m,n)x^n$$

then the m roots of $P_m(x)$ are real, distinct, and nonpositive for all $m = 1, 2, \ldots$.

Proof: It can be easily seen that $P_m(x)$, using (9), may be expressed as

$$\begin{aligned} P_m(x) &= \sum_{n=1}^{m} L_k(m,n)x^n \\ &= \sum_{n=1}^{m} [(nk+m-1)L_k(m-1,n) + kL_k(m-1,n-1)]x^n \\ &= (kx+m-1)P_{m-1}(x) + kx[dP_{m-1}(x)/dx]. \end{aligned} \tag{11}$$

TABLE I

ASSOCIATED LAH NUMBERS, $L_k(m,n)$

$m \backslash n$	1	2	3	4	5
1	k				
2	$k(k+1)$	k^2			
3	$k(k+1)(k+2)$	$3k^2(k+1)$	k^3		
4	$k(k+1)(k+2)(k+3)$	$k^2(k+1)(7k+11)$	$6k^3(k+1)$	k^4	
5	$k(k+1)(k+2)(k+3)(k+4)$	$5k^2(k+1)(k+2)(3k+5)$	$5k^3(k+1)(5k+7)$	$10k^4(k+1)$	k^5

By induction, we find that

$$P_1(x) = kx, \; P_2(x) = kx(kx + k + 1),$$

and

$$P_3(x) = kx[k^2x^2 + 3k(k + 1)x + (k + 1)(k + 2],$$

so that the statement is true for $m = 1$, 2, and 3. For $m > 3$, assume that $P_{m-1}(x)$ has $m - 1$ real, distinct, and nonpositive roots. If we define

$$T_m(x) = e^x x^{m/k} P_m(x), \tag{12}$$

then, since

$$P_m(0) = 0,$$

$T_m(x)$ has exactly the same finite roots as $P_m(x)$, and the identity (11) for $P_m(x)$ gives

$$T_m(x) = kx^{(k+1)/k} dT_{m-1}(x)/dx. \tag{13}$$

By hypothesis, $P_{m-1}(x)$, and hence $T_{m-1}(x)$, has $m - 1$ real, distinct, and nonpositive roots. $T_{m-1}(x)$ also has a root at $-\infty$, and, by Rolle's theorem, between any two roots of $T_{m-1}(x)$, $dT_{m-1}(x)/dx$ will have a root. This places $m - 1$ distinct roots of $T_{m-1}(x)$ on the negative real axis; $x = 0$ is obviously another one, making m altogether. This proves the result by induction.

Thus the above lemma, together with the inequality (10), provides us the following:

Theorem: For $m \geq 3$, $k = 1, 2, \ldots$, and $n = 2, 3, \ldots, m - 1$, the associated Lah numbers $L_k(m,n)$ satisfy the inequality (5).

It may be remarked that, as a consequence of the above result and relation (4), we have the following:

Corollary: For $m \geq 3$, and $n = 2, 3, \ldots, m - 1$, the Lah numbers $L(m,n)$ satisfy the inequality

$$[L(m,n)]^2 > L(m,n + 1)L(m,n - 1). \tag{14}$$

REFERENCES

1. J. C. Ahuja, "Distribution of the Sum of Independent Decapitated Negative Binomial Variables," *Ann. Math. Statist.* 42 (1971):383-384.
2. W. Feller, *An Intorduction to Probability Theory and Its Applications* (New York: Wiley, 1968).
3. G. H. Hardy, J. E. Littlewood, & G. Ploya, *Inequalities* (Cambridge: The University Press, 1952).
4. J. Riordan, *An Introduction to Combinatorial Analysis* (New York: Wiley, 1958).

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