ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$ 

Also $a$ and $b$ designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-400 Proposed by Herta T. Freitag, Roanoke, VA

Let $T_n$ be the $n$th triangular number $n(n+1)/2$. For which positive integers $n$ is $T_1^2 + T_2^2 + \cdots + T_n^2$ an integral multiple of $T_n$?

B-401 Proposed by Gary L. Mullen, Pennsylvania State University, Sharon, PA

Show that $\lim_{n \to \infty} [(n!)^2/(n^2)!] = 0$.

B-402 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Show that $(L_nL_{n+3}, 2L_{n+1}L_{n+2}, 5F_{2n+3})$ is a Pythagorean triple.

B-403 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Let $m = 5^n$. Show that $E_{2m} \equiv -2 \pmod{5m^2}$.

B-404 Proposed by Phil Mana, Albuquerque, NM

Let $x$ be a positive irrational number. Let $a$, $b$, $c$, and $d$ be positive integers with $a/b < x < c/d$. If $a/b < r < x$, with $r$ rational, implies that the denominator of $r$ exceeds $b$, we call $a/b$ a good lower approximation (GLA) for $x$. If $x < r < c/d$, with $r$ rational, implies that the denominator of $r$ exceeds $d$, $a/b$ is a good upper approximation (GUA) for $x$. Find all the GLAs and all the GUAs for $(1 + \sqrt{5})/2$.

B-405 Proposed by Phil Mana, Albuquerque, NM

Prove that for every positive irrational $x$, the GLAs and GUAs for $x$ (as defined in B-404) can be put together to form one sequence $\{p_n/q_n\}$ with

$$p_{n+1}q_n - p_nq_{n+1} = \pm 1$$

for all $n$.  

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Solutions

Complementary Primes

B-376 Proposed by Frank Kocher and Gary L. Mullen, Pennsylvania State University, University Park and Sharon, PA

Find all integers \( n > 3 \) such that \( n - p \) is an odd prime for all odd primes \( p \) less than \( n \).

Solution by Paul S. Bruckman, Concord, CA

Let \( n \) be a solution to the problem, and \( p \) any odd prime less than \( n \). Since \( p \) and \( n - p \) are odd, clearly \( n \) must be even. Hence, \( n \equiv 0, 2, 4 \pmod{6} \). Since \( 4 - 3 = 6 - 5 = 8 - 7 = 1 \) and \( 1 \) is not a prime, it follows that \( n \neq 4, n \neq 6, n \neq 7 \). Hence, \( n \geq 10 \). If \( n \equiv 0 \pmod{6} \), then \( n - 3 \equiv 3 \pmod{6} \), which shows that \( n - 3 \) is composite and \( \geq 9 \). Likewise, if \( n \equiv 2 \pmod{6} \), then \( n - 5 \equiv 3 \pmod{6} \), which shows that \( n - 5 \) is composite and \( \geq 9 \). Finally, if \( n \equiv 4 \pmod{6} \), then \( n - 7 \equiv 3 \pmod{6} \), which is composite, unless \( n = 10 \), in which case \( n - 7 = 3 \), a prime. Hence, \( n = 10 \) is the only possible solution. Since \( 10 - 3 = 7, 10 - 5 = 5, 10 - 7 = 3 \), which are all primes, \( n = 10 \) is indeed the only solution to the problem.

Also solved by Heiko Harborth (W. Germany), Charles Joscelyne, Graham Lord, J. M. Metzger, Bob Prielipp, E. Schmutz & M. Wachtel (Switzerland), Sahib Singh, Rolf Sonntag (W. Germany), Charles W. Trigg, Gregory Wulczyn, and the proposer.

Counting Lattice Points

B-377 Proposed by Paul S. Bruckman, Concord, CA

For all real numbers \( a \geq 1 \) and \( b \geq 1 \), prove that

\[
\sum_{k=1}^{[a]} \left[ b\sqrt{1 - (k/a)^2} \right] = \sum_{k=1}^{[b]} \left[ a\sqrt{1 - (k/b)^2} \right],
\]

where \([x]\) is the greatest integer in \( x \).

Solution by J. M. Metzger, University of North Dakota, Grand Forks, ND

Each sum counts the number of lattice points in the first quadrant of

\[
\frac{x^2}{a} + \frac{y^2}{b} = 1,
\]

the first along the vertical lines, \( x = 1, x = 2, \ldots, x = [a] \), the second along the horizontal lines, \( y = 1, y = 2, \ldots, y = [b] \). The two counts must agree.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

Congruence Mod 3

B-378 Proposed by George Berzsenyi, Laram University, Beaumont, TX

Prove that \( F_{3n+1} + 4^n F_{n+3} \equiv 0 \pmod{3} \) for \( n = 0, 1, 2, \ldots \).
Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, WI

We shall establish that $F_{3k+1} + F_{n+3} \equiv 0 \pmod{3}$ for $n = 0, 1, 2, \ldots$, which is equivalent to the stated result because $4^n \equiv 1 \pmod{3}$ for each nonnegative integer $n$. Clearly the desired result holds when $n = 0$ and when $n = 1$. Assume that $F_{3k+1} + F_{k+3} \equiv 0 \pmod{3}$ and $F_{3k+4} + F_{k+4} \equiv 0 \pmod{3}$, where $k$ is an arbitrary nonnegative integer. Then, by addition,

$$F_{3k+1} + F_{3k+4} + F_{k+5} \equiv 0 \pmod{3}.$$  

But

$$6F_{3k+2} + 4F_{3k+1} + F_{3k+4} = F_{3k+7}$$

so

$$F_{3k+1} + F_{3k+4} \equiv F_{3k+7} \pmod{3}.$$  

Hence

$$F_{3k+7} + F_{k+5} \equiv 0 \pmod{3}$$

and our proof is complete by mathematical induction.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, Sahib Singh, Gregory Wulczyn, and the proposer.

Congruence Mod 5

B-379 Proposed by Herta T. Freitag, Roanoke, VA

Prove that $F_{2n} \equiv n(-1)^{n+1} \pmod{5}$ for all nonnegative integers $n$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, WI

Clearly the desired result holds when $n = 0$ and when $n = 1$. Assume that $F_{2k} \equiv k(-1)^{k+1} \pmod{5}$ and $F_{2k+2} \equiv (k+1)(-1)^{k+2} \pmod{5}$, where $k$ is an arbitrary nonnegative integer. Then, since

$$F_{2k+4} = 3F_{2k+2} - F_{2k},$$

$$F_{2k+4} \equiv (3k + 3)(-1)^{k+2} - k(-1)^{k+1} \pmod{5}$$

$$\equiv (-1)^{k+2}(4k + 3) \pmod{5}$$

$$\equiv (k + 2)(-1)^{k+3} \pmod{5}.$$  

Our solution is now complete by mathematical induction.

Also solved by Paul S. Bruckman, Charles Joscelynne, Graham Lord, Sahib Singh, Gregory Wulczyn, and the proposer.

Binomial Convolution

B-380 Proposed by Dan Zwillinger, Cambridge, MA

Let $a$, $b$, and $c$ be nonnegative integers. Prove that

$$\sum_{k=1}^{n} \binom{k+a-1}{a} \binom{n-k+b-c}{b} = \binom{n+a+b-c}{a+b+1}.$$  

Here $\binom{m}{r} = 0$ if $m < r$. 


Solution by Phil Mana, Albuquerque, NM

For every nonnegative integer $d$, the Maclaurin series for $(1 - x)^{-d-1}$ is

$$
\sum_{n=0}^{\infty} \binom{n+d}{d} x^n.
$$

Then

$$(1 - x)^{-\alpha-1}(1 - x)^{-\beta-1} = (1 - x)^{-\alpha-\beta-2},$$

$$
\sum_{i=0}^{\infty} \binom{i+\alpha}{\alpha} x^i \cdot \sum_{j=0}^{\infty} \binom{j+\beta}{\beta} x^j = \sum_{n=0}^{\infty} \binom{n+\alpha+\beta+1}{\alpha+\beta+1} x^n.
$$

Equating coefficients of $x^{n-\alpha-1}$ on both sides, one has

$$
\sum_{k=1}^{n-\alpha} \binom{k-1+\alpha}{\alpha} \binom{n-\alpha-k+\beta}{\beta} = \binom{n-\alpha+\alpha+\beta}{\alpha+\beta+1}
$$

The upper limit $n - \alpha$ for the sum here can be replaced by $n$, since any terms for $n - \alpha < k \leq n$ will vanish using the convention that $\binom{m}{r} = 0$ for $m < r$. This gives the desired result.

Also solved by Paul S. Bruckman, Bob Prielipp & N. J. Kuenzl, A. G. Shannon, and the proposer.

Generating Function

B-381 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA

Let $a_{2n} = F_n^2$ and $a_{2n+1} = F_{n+1} F_{n+2}$. Find the rational function that has

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

as its Maclaurin series.

Solution by Sahib Singh, Clarion State College, Clarion, PA

By the result $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$, we get the Maclaurin series as:

$$
F_1^2 + F_2^2 x (1 + x^2 + x^4 + \cdots) + F_3^2 x^2 (1 + x^2 + x^6 + \cdots) + \cdots
$$

$$
= F_1^2 \left(1 + \frac{x}{1 - x^2}\right) + F_2^2 \frac{x^2}{1 - x^2} \left(1 + \frac{x}{1 - x^2}\right) + F_3^2 \frac{x^4}{1 - x^2} \left(1 + \frac{x}{1 - x^2}\right) + \cdots
$$

$$
= \frac{1 + x - x^2}{1 - x^2} \left[ F_1^2 + F_2^2 x^2 + F_3^2 x^4 + F_4^2 x^6 + \cdots \right].
$$

Using $F_n^2 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2$, the above becomes
\[
\frac{1 + x - x^2}{1 - x^2} \cdot \frac{1}{(a - b)^2} \left[(a^2 + a^5 x^2 + a^6 x^4 + \cdots) + (b^2 + b^5 x^2 + b^6 x^4 + \cdots) - 2ab(1 + ab^2 + a^2 b^2 x^4 + \cdots)\right] \\
= \left(\frac{1 + x - x^2}{1 - x^2}\right) \cdot \frac{1}{(a - b)^2} \left[\frac{a^2}{1 - a^2 x^2} + \frac{b^2}{1 - b^2 x^2} - \frac{2ab}{1 - abx^2}\right],
\]
which simplifies to
\[
\left(\frac{1 + x - x^2}{1 - x^2}\right) \left(\frac{(1 - x^2)}{(1 + x^2)(1 - 3x^2 + x^6)}\right) = \frac{1 + x - x^2}{(1 + x^2)(1 - 3x^2 + x^4)}. 
\]

Also solved by Paul S. Bruckman, R. Garfield, John W. Vogel, and the proposer.

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ERRATA

The following errors have been noted:

Volume 16, No. 5 (October 1978), p. 407 [J. A. H. Hunter's "Congruent Primes of Form \((8n+1)\)"]. The equations presented in the second line of the article should read
\[ x^2 - eY^2 = z^2, \text{ and } x^2 + eY^2 = z^2. \]

Volume 17, No. 1 (February 1979), p. 84 (A. P. Hillman & V. E. Hoggatt, Jr.'s "Nearly Linear Functions"). Equation (1) should read
\[
(1) \quad C^\prime \cdot H - C \cdot H = \sum_{t=1}^{k} (c_t^\prime - c_t) h_t \geq h_k - \sum_{t=1}^{k-1} c_t h_t. 
\]
The second line of the proof of Lemma 7 should read
The hypothesis \(E \cdot E^\prime = 0\) implies . . .

In the proof of Theorem 1, Equation (10) should read
\[
(10) \quad b_j(n) = C_n^m \cdot H_j - C_{n-1} \cdot H_j. 
\]

(Kindness of Margaret Owens)