We can assume that $a' \not\equiv 0 \pmod{p}$ since, by hypothesis, $a' \equiv 0 \pmod{p}$ holds only for finitely many primes p. Then if $w_1 - a'w_0 \not\equiv 0 \pmod{p}$, $w_n \equiv 0$ when

$$n \equiv -a'w_0/(w_1 - a'w_0) \pmod{p}.$$

If $w_1 - a'w_0 \equiv 0 \pmod{p}$ for almost all primes p, then $w_1 = a'w_0$. Hence, by (13),

$$w_n = (a')^n w_0 = \alpha^n w_0.$$

In this case, the only primes which are divisors of the recurrence are those primes which divide $a'w_0$. Note that if the hypotheses of (iii) hold, then the only recurrences not having almost all primes as divisors are those that are multiples of translations of the Lucas sequence $\{v_n\}$.

(iv) Since

$$\alpha^{n+2} = a\alpha^{n+1} + b\alpha^n$$

and

$$\beta^{n+2} = a\beta^{n+1} + b\beta^n,$$

it follows that either the terms of the recurrence $\{w_n\}$ are of the form $\{\alpha^n w_0\}$ or they are of the form $\{\beta^n w_0\}$. The result is now easily obtained.

To conclude, we note that as a counterpoise to Theorem 1, which states that essentially only one class of recurrences has almost all primes as divisors, there is the following theorem by Morgan Ward [3]. It states that, in general, every recurrence has an infinite number of prime divisors.

Theorem 3 (Ward): In the recurrence $\{w_n\}$ with parameters a and b, suppose that $b \neq 0$, $w_1 \neq \alpha w_0$, and $w_1 \neq \beta w_0$. Then if α/β is not a root of unity, the recurrence $\{w_n\}$ has an infinite number of prime divisors.

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NOTE ON A TETRANACCI ALTERNATIVE TO BODE'S LAW

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Bode's law is an empirical approximation to the mean distances of the planets from the Sun; it arises from a simply-generated sequence of integers. Announced in 1772 by Titius and later appropriated by Bode, it has played an important role in the exploration of the Solar System [1].

The Bode numbers are defined by

$$B_1 = 4$$

$$B_n = 2^{n-2} \times 3 + 4,\ n = 2,\ \ldots$$

Then the quantities $0.1B_n$, $n = 1, \ldots, 10$ represent the mean distances of the nine planets and the asteroid belt from the Sun in terms of the Earth's distance.

In view of the numerical explorations reported in [2], [3], and [4], it seems plausible to look for improvements to Bode's law among the Multinacci sequences and, indeed, the Tribonacci, Tetranacci, Pentanacci, and Hexanacci numbers are suited to this task. The Tetranacci numbers provide the best fit, slightly superior to the original Bode solution.

The Tetranacci numbers are defined by the recurrence

$$T_1, \ldots, T_4 = 1$$

$$T_n = \sum_{i=1}^{4} T_{n-i}, \; n = 5, \ldots$$

The alternative Bode numbers are then given by

$$\tilde{B}_n = T_{n+3} + 3, \; n = 1, \ldots$$

The quantities $0.1\tilde{B}_n$ can then be compared with their Bode counterparts. See the accompanying table.

Planet	Actual Distance	Bode	Tetranacci
Mercury	0.39	0.40	0.40
Venus	0.72	0.70	0.70
Earth	1.00	1.00	1.00
Mars	1.52	1.60	1.60
(asteriods)	2.70	2.80	2.80
Jupiter	5.20	5.20	5.20
Saturn	9.54	10.00	9.70
Uranus	19.18	19.60	18.40
Neptune	30.06	38.80	35.20
Pluto	39.44	77.20	67.60

It can be seen that the fits are poor for Neptune and bad for Pluto. However, the Tetranacci alternative is somewhat better in both cases.

No rigorous dynamical explanation is apparent for the Bode or Tetranacci representations. They are either numerical coincidences, as the result in [5] indicates, or, if they contain physical information, may simply illustrate that the period of revolution of a planet is strongly a function of the periods of nearby planets. This conjecture arises from the Kepler relation $(\text{distance})^3 \propto (\text{period})^2$ and the fact that period relationships are often important in determining the state of a dynamical system.

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REFLECTIONS ACROSS TWO AND THREE GLASS PLATES

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1. INTRODUCTION

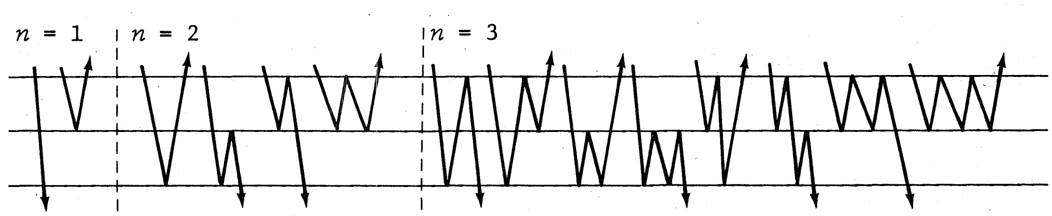
That reflections of light rays within two glass plates can be expressed in terms of the Fibonacci numbers is well known [Moser, 1]. In fact, if one starts with a single light ray and if the surfaces of the glass plates are half-mirrors such that they both transmit and reflect light, the number of possible paths through the glass plates with n reflections is F_{n+2}. Hoggatt and Junge [2] have increased the number of glass plates, deriving matrix equations to relate the number of distinct reflected paths to the number of reflections and examining sequences of polynomials arising from the characteristic equations of these matrices.

Here, we have arranged the counting of the reflections across the two glass plates in a fresh manner, fixing our attention upon the number of paths of a fixed length. One result is a physical interpretation of the compositions of an integer using 1's and 2's (see [3], [4], [5]). The problem is extended to three glass plates with geometric and matrix derivations for counting reflection paths of different types as well as analyses of the numerical arrays themselves which arise in the counting processes. We have counted reflections in paths of fixed length for regular and for bent reflections, finding powers of two, Fibonacci numbers and convolutions, and Pell numbers.

2. PROBLEM I

Consider the compositions of an even integer $2n$ into ones and twos as represented by the possible paths of length $2n$ taken in reflections of a light ray in two glass plates.

REFLECTIONS OF A LIGHT RAY IN PATHS OF LENGTH $2n$



For a path length of 2, there are 2 possible paths and one reflection; for a path length of 4, 4 possible paths and 8 reflections; for a path length of 6,