For example, by $m = 2$, we find again formula (1) under the term

$$(x_1 + x_2)^n = \sum_{k=1}^{n} S_k \binom{2n-k-1}{n-1} \left(\frac{\sigma_2}{\sigma_1}\right)^{n-k}.$$

For three variables, x_1, x_2, x_3, $m = 3$, we have ($\nu = \nu_2$):

$$(x_1 + x_2 + x_3)^n = \sum_{k=1}^{n} \left\{ S_k \sum_{0 \le \nu \le \frac{n-k}{2}} (-1)^\nu \frac{(2n-k-1-\nu)!}{(n-1)!\,\nu!\,(n-k-2\nu)!} \left(\frac{\sigma_2}{\sigma_1}\right)^{n-k-2\nu} \left(\frac{\sigma_3}{\sigma_1}\right)^{\nu} \right\}.$$

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WHICH SECOND-ORDER LINEAR INTEGRAL RECURRENCES HAVE ALMOST ALL PRIMES AS DIVISORS?

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This paper will prove that essentially only the obvious recurrences have almost all primes as divisors. An integer n is a divisor of a recurrence if n divides some term of the recurrence. In this paper, "almost all primes" will be taken interchangeably to mean either all but finitely many primes or all but for a set of Dirichlet density zero in the set of primes. In the context of this paper, the two concepts become synonymous due to the Frobenius density theorem. Our paper relies on a result of A. Schinzel [2], whose paper uses "almost all" in the same sense.

Let $\{w_n\}$ be a recurrence defined by the recursion relation

(1) $$w_{n+2} = aw_{n+1} + bw_n$$

where a, b, and the initial terms w_0, w_1 are all integers. We will call a and b the parameters of the recurrence. Associated with the recurrence (1) is its characteristic polynomial

(2) $$x^2 - ax - b = 0,$$

with roots α and β, where $\alpha + \beta = a$ and $\alpha\beta = -b$.

Let

$$D = (\alpha - \beta)^2 = a^2 + 4b$$

be the discriminant of this polynomial.

In general, if $D \neq 0$,

(3) $$w_n = c_1\alpha^n + c_2\beta^n,$$

where

(4) $$c_1 = (w_1 - w_0\beta)/(\alpha - \beta)$$

and

(5) $$c_2 = (w_0\alpha - w_1)/(\alpha - \beta).$$

We allow n to be negative in (3), though then w_n is rational but not necessarily integral.

There are two special recurrences with parameters a and b which we will refer to later. They are the Primary Recurrence (PR) $\{u_n\}$ with initial terms $u_0 = 0$, $u_1 = 1$ and the Lucas sequence $\{v_n\}$ with initial terms $v_0 = 2$ and $v_1 = a$. By (4) and (5) we see that the nth term of the PR is

(6) $$u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$$

and the nth term of the Lucas sequence is

(7) $$v_n = \alpha^n + \beta^n.$$

The following lemma will help give us a partial answer to the problem of determining those recurrences which have almost all primes as divisors.

Lemma 1: Let $\{w_n\}$ be a recurrence with parameters a and b. Let p be a prime. If $b \not\equiv 0 \pmod{p}$, then $\{w_n\}$ is purely periodic modulo p.

Proof: First, if a pair of consecutive terms (w_n, w_{n+1}) is given, the recurrence $\{w_n\}$ is completely determined from that point on by the recursion relation. Now, a pair of consecutive terms (w_m, w_{m+1}) must repeat (mod p) since only p^2 pairs of terms are possible (mod p). Suppose (w_m, w_{m+1}) is the first pair of terms to repeat (mod p) with $m \neq 0$. But then

$$bw_{m-1} = w_{m+1} - aw_m$$

by the recursion relation. Hence,

$$w_{m-1} \equiv b^{-1}(w_{m+1} - aw_m) \pmod{p}.$$

Thus, w_{m-1} is now determined uniquely (mod p) and the pair $(w_{m-1}, w\ )$ repeats (mod p) which is a contradiction. Therefore, $m = 0$ and the sequence is purely periodic modulo p.

Thus, we now have at least a partial answer to the question of our title. The PR $\{u_n\}$ clearly satisfies our problem since any prime divides the initial term $u_0 = 0$. Further, any multiple of a translation of this sequence also works. The sequence $\{w_n\}$, where $w_0 = ru_{-n}$, $w_1 = ru_{-n+1}$ with r rational and $n \geq 0$ clearly has 0 as a term. Moreover, by our previous result, Lemma 1, if $p \nmid b$, then p divides some term of $\{w_n\}$, where $w_0 = ru_n$, $w_1 = ru_{n+1}$ with r rational and $n \geq 0$. Clearly, there are only finitely many primes p dividing b. We shall show that these are essentially the only such recurrences satisfying our problem. This is expressed in the following main theorem of our paper.

Theorem 1: Consider the recurrence $\{w_n\}$ with parameters a and b. Suppose

$$b \neq 0,\ D \neq 0,\ w_1 \neq \alpha w_0,\ \text{and}\ w_1 \neq \beta w_0.$$

Then almost all primes are divisors of the recurrence $\{w_n\}$ if and only if

$$w_0 = ru_n,\ w_1 = ru_{n+1}$$

for some rational r and integer n, not necessarily positive.

We will now explore how far we can go towards proving our main theorem using just elementary and well-known results of number theory.

Theorem 2: Consider the recurrence $\{w_n\}$ with parameters a and b. Suppose that neither $w_1^2 - w_0w_2$ nor $(-b)(w_1^2 - w_0w_2)$ is a perfect square. Then, there exists a set of primes of positive density that does not contain any divisors of $\{w_n\}$.

Proof: It can be proved by induction that

$$w_n^2 - w_{n-1}w_{n+1} = (w_1^2 - w_0w_2)(-b)^{n-1}. \tag{8}$$

By the law of quadratic reciprocity, the Chinese remainder theorem, and Dirichlet's theorem on the infinitude of primes in arithmetic progressions, it can be shown that there exists a set of primes p of positive density such that

$$(-b/p) = 1 \quad \text{and} \quad (w_1^2 - w_0w_2/p) = -1.$$

We suppress the details. Now suppose that p divides some term w_{n-1}. Then

$$w_n^2 - 0 \equiv (w_1^2 - w_0w_2)(-b)^{n-1} \pmod{p}.$$

But

$$(w_n^2/p) = 1$$

and

$$\left((w_1^2 - w_0w_2)(-b)^{n-1}/p\right) = (1)(-1) = -1.$$

This is a contradiction and the theorem follows.

Unfortunately, there are recurrences which are not multiples of translations of PRs and which do not satisfy the hypothesis of Theorem 2. For example, consider the recurrence $\{w_n\}$ with parameters $a = 3$, $b = 5$, and initial terms 5, 21, 88, 369. Then

$$w_1^2 - w_0w_2 = 1$$

and the conditions of Theorem 2 are not met. However, it is easily seen that this recurrence is not a multiple of a translation of the PR with parameters 3 and 5.

To prove our main theorem, we will need a more powerful result.

Lemma 2: Let L be an algebraic number field. If λ and θ are nonzero elements of L and the congruence

$$\lambda^x \equiv \theta \pmod{P}$$

is solvable in rational integers for almost all prime ideals P of L, then the corresponding equation

$$\lambda^x = \theta$$

is solvable for a fixed rational integer.

Proof: This is a special case of Theorem 2 of A. Schinzel's paper [2].

Before going on, we will need three technical lemmas.

Lemma 3: In the PR $\{u_n\}$ with parameters a and b, suppose that $b \neq 0$. Then

$$u_{-n} = (-1)^{n+1}(u_n/b^n) \quad \text{for } n \geq 0.$$

Proof: Use induction on n.

Lemma 4: Consider the PR $\{u_n\}$ with parameters a and b. Then

$$\alpha_n = bu_{n-1} + u_n\alpha$$

and

$$\beta_n = bu_{n-1} + u_n\beta$$

where $n \geq 0$.

Proof: Notice that

(9) $$\alpha^{n+2} = a\alpha^{n+1} + b\alpha^n$$

and

(10) $$\beta^{n+2} = a\beta^{n+1} + b\beta^n.$$

Now use induction on n and the recursion relations (9) and (10).

Lemma 5: In the recurrence $\{w_n\}$ with parameters a and b, suppose that

$$D \neq 0,\ b \neq 0,\ w_1 \neq \alpha w_0,\ \text{and}\ w_1 \neq \beta w_0.$$

Let $\gamma = w_1 - w_0\alpha$ and $\delta = w_1 - w_0\beta$ be the roots of the quadratic equation

$$x^2 - (2w_1 - aw_0)x - (bw_0^2 + aw_0w_1 - w_1^2) = 0.$$

Then

$$\gamma/\delta = (\alpha/\beta)^n$$

for some rational integer n, not necessarily positive, if and only if

$$w_0 = ru_{-n},\ w_1 = ru_{-n+1}$$

for some rational number r.

Proof: First we will prove necessity. Suppose that

$$\gamma/\delta = (\alpha/\beta)^n.$$

By hypothesis none of α, β, γ, or δ is equal to 0. Then $\gamma = m\alpha^n$ and $\delta = m\beta^n$ for some element m of the algebraic number field $K = Q(\sqrt{D})$. We now claim that m is a rational number. Let t_k be the kth term of the PR with parameters $2w_1 - aw_0$ and $bw_0^2 + aw_0w_1 - w_1^2$. Then

$$t_k = (\gamma^k - \delta^k)/(\gamma - \delta).$$

In particular,

$$t_2 = 2w_1 - aw_0 = (m^2\alpha^{2n} - m^2\beta^{2n})/(m\alpha^n - m\beta^n)$$
$$= m(\alpha^n + \beta^n) = mv_n,$$

where v_n is the nth term of the Lucas sequence with parameters a and b. Hence

$$m = (2w_1 - aw_0)/v_n$$

is a rational number. Now remember that

$$\gamma = w_1 - w_0\alpha = m\alpha^n \quad \text{and} \quad \delta = w_1 - w_0\beta = m\beta^n.$$

By Lemma 4, we can express α^n and β^n in terms of u_{n-1}, u_n, α, and β. Now γ and δ are already expressed in terms of w_0, w_1, α, and β. We can thus solve for w_0, w_1 in terms of α, β, u_{n-1}, and u_n. We now use Lemma 3 to express u_{-n} in terms of u_n. If n is positive, we obtain

(11) $$w_0 = [(-1)^n mb^n]u_{-n},\quad w_1 = [(-1)^n mb^n]u_{-n+1}.$$

If n is negative or zero, we obtain

(12) $$w_0 = mu_{-n}, \quad w_1 = mu_{-n+1}$$

as required. We have now proved necessity. To prove sufficiency, we simply reverse our steps in the proof so far.

We are now ready for the proof of our main theorem.

Proof of Theorem 1: We have already shown the sufficiency of the theorem in our remarks following Lemma 1. To prove necessity, suppose that for almost all primes p there exists a rational integer n such that $p|w_n$. Then by (3),

$$w_n = c_1\alpha^n + c_2\beta^n \equiv 0 \pmod p$$

is satisfiable for some integral n for almost all rational primes p. In the algebraic number field $K = Q(\sqrt{D})$, we thus have

$$c_1\alpha^n + c_2\beta^n \equiv 0 \pmod P$$

for the prime ideals P dividing (p) in K. Thus,

$$(\alpha/\beta)^n \equiv -c_2/c_1 \equiv \gamma/\delta \pmod P$$

by the definition c_1, c_2, γ, and δ. Consequently,

$$\gamma/\delta \equiv (\alpha/\beta)^x \pmod P$$

is solvable for almost all prime ideals P in K. Hence, by Lemma 2,

$$\gamma/\delta = (\alpha/\beta)^n$$

for some rational integer n. Therefore, by Lemma 5,

$$w_0 = ru_{-n}, \quad w_1 = ru_{-n+1}$$

for some rational number r and we are done.

For completeness, the next theorem will answer the question of the title for those recurrences excluded by the hypothesis of Theorem 1.

Theorem 3: In the recurrence $\{w_n\}$ with parameters a and b, suppose that

$$(w_0,w_1) = (0,0),\ b = 0,\ D = 0,\ w_1 = \alpha w_0, \text{ or } w_1 = \beta w_0.$$

Let p denote a rational prime.

(i) If $w_0 = 0$ and $w_1 = 0$, then $p|w_n$ for all n regardless of a and b. Note that in this case, the recurrence $\{w_n\}$ is a multiple of the PR $\{u_n\}$.

(ii) If $b = 0$ and $(w_0,w_1) \neq (0,0)$, then the recurrence $\{w_n\}$ has almost all primes as divisors only in the following cases:

(a) $b = 0$, $a \neq 0$, $w_0 = 0$, and $w_1 \neq 0$. Then $p|w_0$ for all primes p and $p\nmid w_n$, $n \geq 1$, if $p\nmid aw_1$. Clearly, in this case the recurrence is a multiple of the PR $\{u_n\}$.

(b) $b = 0$, $a \neq 0$, $w_0 \neq 0$, and $w_1 = 0$. Then $w_n = 0$ for $n \geq 1$ and $p|w_n$ for all p if $n \geq 1$.

(c) $b = 0$, $a = 0$. Then $p|w_n$ for all p if $n \geq 2$.

(iii) Suppose $b = 0$, $(w_0,w_1) \neq (0,0)$, $a \neq 0$, and $b \neq 0$. Then the recurrence $\{w_n\}$ has almost all primes p as divisors if and only if $w_1 \neq (a/2)w_0$.

(iv) Suppose that $w_1 = \alpha w_0$ or $w_1 = \beta w_0$. Further, suppose that D is a perfect square, $w_0 \neq 0$, and $b \neq 0$. Then almost all primes are *not* divisors of the recurrence $\{w_n\}$. Moreover, $p\nmid w_n$ for any n if $p\nmid w_1$.

Proof: (i) and (ii) can be proved by direct verification.

(iii) Let $a' = a/2$. It can be shown by induction that

(13) $$w_n = (a')^{n-1}(a'w_0 + (w_1 - aw_0)n).$$

We can assume that $a' \not\equiv 0 \pmod{p}$ since, by hypothesis, $a' \equiv 0 \pmod{p}$ holds only for finitely many primes p. Then if $w_1 - a'w_0 \not\equiv 0 \pmod{p}$, $w_n \equiv 0$ when

$$n \equiv -a'w_0/(w_1 - a'w_0) \pmod{p}.$$

If $w_1 - a'w_0 \equiv 0 \pmod{p}$ for almost all primes p, then $w_1 = a'w_0$. Hence, by (13),

$$w_n = (a')^n w_0 = \alpha^n w_0.$$

In this case, the only primes which are divisors of the recurrence are those primes which divide $a'w_0$. Note that if the hypotheses of (iii) hold, then the only recurrences not having almost all primes as divisors are those that are multiples of translations of the Lucas sequence $\{v_n\}$.

(iv) Since

$$\alpha^{n+2} = a\alpha^{n+1} + b\alpha^n$$

and

$$\beta^{n+2} = a\beta^{n+1} + b\beta^n,$$

it follows that either the terms of the recurrence $\{w_n\}$ are of the form $\{\alpha^n w_0\}$ or they are of the form $\{\beta^n w_0\}$. The result is now easily obtained.

To conclude, we note that as a counterpoise to Theorem 1, which states that essentially only one class of recurrences has almost all primes as divisors, there is the following theorem by Morgan Ward [3]. It states that, in general, every recurrence has an infinite number of prime divisors.

Theorem 3 (Ward): In the recurrence $\{w_n\}$ with parameters a and b, suppose that $b \neq 0$, $w_1 \neq \alpha w_0$, and $w_1 \neq \beta w_0$. Then if α/β is not a root of unity, the recurrence $\{w_n\}$ has an infinite number of prime divisors.

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NOTE ON A TETRANACCI ALTERNATIVE TO BODE'S LAW

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Bode's law is an empirical approximation to the mean distances of the planets from the Sun; it arises from a simply-generated sequence of integers. Announced in 1772 by Titius and later appropriated by Bode, it has played an important role in the exploration of the Solar System [1].

The Bode numbers are defined by

$$B_1 = 4$$

$$B_n = 2^{n-2} \times 3 + 4, \; n = 2, \ldots$$