(3.9)
$$f_{j+k\pm 1} = f_{j\pm 1}f_{k\pm 1} + f_j f_k;$$
$$f_{j+k} = f_{j+1}f_k + f_j f_{k+1}.$$

From the above, or from $F(np) = F(n)^p$, we are also able to obtain other familiar expressions such as:

(3.10)
$$f_{2n\pm 1} = f_n^2 + f_{n\pm 1}^2;$$
$$\frac{f_{2n}}{f_n} = f_{n+1} + f_{n-1}$$
$$f_{3n} = f_{n+1}^3 + f_n^3 - f_{n-1}^3;$$
$$\frac{f_{3n}}{f_n} = 2f_{n+1}^2 + f_n^2 + f_{n+1}f_{n-1}.$$

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A NOTE ON BASIC M-TUPLES

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Definition 1: A set of integers $\{b_i\}_{i\geq 1}$ will be called a base for the set of all integers, whenever every integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{\infty} a_i b_i, \text{ where } a_i = 0 \text{ or } 1 \quad\text{and}\quad \sum_{i=1}^{\infty} a_i < \infty.$$

Now, a sequence $\{d_i\}_{i\geq 1}$ of odd numbers will be called basic whenever the sequence $\{d_i 2^{i-1}\}_{i\geq 1}$ is a base. If the sequence $\{d_i\}_{i\geq 1}$ of odd integers is such that $d_{i+s} = d_i$ for all i's, then the sequence is said to be periodic mod s and is denoted by $\{d_1, d_2, d_3, \ldots, d_s\}$. In reference [2], I have obtained some results concerning nonbasic sequence with periodicity mod 3 or nonbasic triples. In this paper, we are concerned with basic sequence.

Theorem 1: A necessary and sufficient condition for the sequence $\{d_i\}_{i\geq 1}$ of odd integers, which is periodic mod s, to be basic is that

$$0 = \sum_{i=1}^{m} a_i 2^{i-1} d_i \equiv 0 \pmod{2^{ns} - 1}$$

is impossible for $n \geq 1$ and $a_i = 0$ or 1 for all $i \geq 1$.

Proof: This is proved in reference [1].

Theorem 2: The m-tuple

$$\left\{2^{mk+1} - 1, -1, -1, \ldots, -1\right\}$$

is a basic sequence where k and m are integers with $k \geq 1$ and $m \geq 2$.

Proof: Suppose that the given m-tuple is not basic. Then (1) of Theorem 1.8 holds for some integers $n \geq 1$. Then there exist integers $a_i, b_i, \ldots, r_i$ for $0 \leq k \leq n - 1$ such that

$$\begin{aligned}(1)\quad & (2^{mk+1} - 1)a_0 - 2b_0 - 2^2c_0 - \cdots - 2^{m-1}r_0 + (2^{mk+1} - 1)2^m a_1 \\ & -2^{m+1}b_1 - 2^{m+2}c_1 - \cdots - 2^{2m-1}r_1 + (2^{mk+1} - 1)2^{2m}a_2 - 2^{2m+1}b_2 \\ & -2^{2m+2}c_2 - \cdots - 2^{3m-1}r_2 + \cdots + (2^{mk+1} - 1)2^{mn-m}a_{n-1} \\ & -2^{mn-m+1}b_{n-1} - 2^{mn-m+2}c_{n-1} - \cdots - 2^{mn-1}r_{n-1} \equiv 0 \pmod{2^{mn} - 1}.\end{aligned}$$

Collecting terms in the above congruence, we obtain

$$\begin{aligned}& (2 \cdot 2^{mk} - 1)(a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) \\ & -2(b_0 + 2^m b_1 + \cdots + 2^{mn-m}b_{n-1}) - 2^2(c_0 - 2^m c_1 + 2^{2m}c_2 + \cdots \\ & + 2^{mn-m}c_{n-1}) - \cdots - 2^{m-1}(r_0 + 2^m r_1 + \cdots + 2^{mn-m}r_{n-1}) \equiv 0 \pmod{2^m - 1} \\ & \qquad - (a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) \\ & + 2(2^{mk}a_0 + 2^{mk+m}a_1 + 2^{mk+2m}a_2 + \cdots + 2^{mk+mn-m}a_{n-1} \\ & -b_0 - 2^m b_1 - \cdots - 2^{mn-m}b_{n-1} - 2c_0 - 2^{m+1}c_1 - \cdots \\ & - 2^{mn-m+1}c_{n-1} - \cdots - 2^{m-2}r_0 - 2^{2m-2}r_1 - \cdots - 2^{mn-2}r_{n-1}) \\ & \equiv 0 \pmod{2^m - 1}\end{aligned}$$

which can be put in the form

$$\begin{aligned}& -(a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) + 2\{2^{mk}(a_0 - b_k - 2c_k - \cdots \\ & - 2^{m-2}r_k) + 2^{m(k+1)}(a_1 - b_{k+1} - 2c_{k+1} - \cdots - 2^{m-2}r_{k+1}) + \cdots \\ & + 2^{m(n-1)}(a_{n-1-k} - b_{n-1} - 2c_{n-1} - \cdots - 2^{m-2}r_{n-1}) \\ & + 2(2^{mn}a_{n-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0) + \cdots \\ & + 2^{m(k-1)}(2^{mn}a_{n-1} - b_{k-1} - 2c_{k-1} - \cdots - 2^{m-2}r_{k-1}) \\ & + (2^{mn}a_{n-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0) + \cdots\end{aligned}$$

$$+ 2^{m(k-1)}(2^{mn}a_{n-1} - b_{k-1} - 2c_{k-1} - \cdots - 2^{m-2}r_{n-1})\}$$
$$\equiv 0 \pmod{2^{mn}-1}.$$

Now define $a_{n-i} = a_{-i}$ for $1 \le i \le k$, and let

$$\begin{aligned} Q = {} & -(a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) + 2\{2^{mk}(a_0 - b_k - 2c_k \\ & - \cdots - 2^{m-2}r_k) + 2^{m(k+1)}(a_1 - b_{k+1} - 2c_{k+1} - \cdots - 2^{m-2}r_{k+1}) \\ & + \cdots + 2^{m(n-1)}(a_{n-1-k} - b_{n-1} - 2c_{n-1} - \cdots - 2^{m-2}r_{n-1}) \\ & + (a_{-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0) + \cdots + 2^{m(k-1)}(a_{-1} - b_{k-1} \\ & - 2c_{k-1} - \cdots - 2^{m-2}r_{k-1})\} \equiv 0 \pmod{2^{mn}-1}. \end{aligned} \tag{3}$$

Rearranging terms in (3), we obtain

$$\begin{aligned} Q = {} & \{-a_0 + 2(a_{-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0)\} + 2^m\{-a_1 + 2(a_{-k+1} \\ & - b_1 - 2c_1 - \cdots - 2^{m-2}r_1) + \cdots + 2^{m(k-1)} -a_{k-1} + 2(a_{-1} - b_{k-1} \\ & - 2c_{k-1} - \cdots - 2^{m-2}r_{k-1}) + 2^{mk} -a_k + 2(a_0 - b_k - 2c_k \\ & - \cdots - 2^{m-2}r_k)\} + \cdots + 2^{m(n-1)}\{-a_{n-1} + 2(a_{n-1-k} - b_{n-1} - 2c_{n-1} \\ & - \cdots - 2^{m-2}r_{n-1})\} \equiv 0 \pmod{2^{mn}-1}. \end{aligned} \tag{4}$$

Taking absolute values and using the triangle inequality, we obtain

$$\begin{aligned} |Q| & \le (2^m - 1) + 2^m(2^m - 1) + 2^{2m}(2^m - 1) + \cdots + 2^{m(n-1)}(2^m - 1) \\ & = (2^m - 1) + (2^{2m} - 2^m) + (2^{3m} - 2^{2m}) + \cdots + (2^{mn} - 2^{m(n-1)}) \\ & = 2^{mn} - 1. \end{aligned}$$

Now, $|Q| = 2^{mn} - 1$, provided

$$-a_i + 2(a_{-k+i} - b_i - 2c_i - \cdots - 2^{m-2}r_i) = 2^m - 1$$

for all i with $0 \le i \le n - 1$. But this clearly implies that

$$a_i = 1,\ a_{-k+i} = 0, \text{ and } b_i = c_i = \cdots = r_i = 1 \quad \text{for all } i.$$

Since the first two equalities are clearly contradictory, it follows that we must have $Q = 0$ and hence

$$-a_i = 2(a_{-k+i} - b_i - 2c_i - \cdots - 2^{m-2}r_i), \tag{5}$$

and yet $a_i = 0$ or $a_i = 1$ for all i. Since the right-hand side of (5) is divisible by 2, it follows that $r_i = 0$ for all i. Thus,

$$0 = 2(a_{-k+i} - b_i - 2c_i - \cdots - 2^{m-2}r_i) \tag{5'}$$

or

(6) $$a_{-k+i} - b_i = 2c_i + \cdots + 2^{m-2}r_i \text{ for all } i.$$

Possibilities for $a_{-k+i} - b_i$ are 0, 1, and -1. But the right-hand side of (6) is divisible by 2. Hence, we must have that $a_{-k+i} - b_i = 0$ for all i. Since $a_{-k+i} = 0$ for all i, this implies that $b_i = 0$ for all i and hence that $c_i = 0, \ldots, r_i = 0$ for all i. But since this contradicts Theorem 1.8, it follows that the m-tuple $2^{mk+1} - 1, -1, -1, \ldots, -1$ is basic as claimed.

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PYTHAGOREAN TRIPLES AND TRIANGULAR NUMBERS

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1. INTRODUCTION

In [4] W. Sierpiński proves that there are an infinite number of Pythagorean triples in which two members are triangular and the hypotenuse is an integer. [A number T_n is triangular if T_n is of the form $T_n = n(n+1)/2$ for some integer n. A Pythagorean triple is a set of three integers x, y, z such that $x^2 + y^2 = z^2$.] Further, Sierpinski gives an example due to Zarankiewicz,

$$T_{132} = 8778, \quad T_{143} = 10296, \quad \text{and} \quad T_{164} = 13530,$$

in which every member of the Pythagorean triple is triangular. He states that this is the only known nontrivial example of this phenomenon, and that it is not known whether the number of such triples is finite or infinite.

This paper will give some partial results related to the above problem. In particular, we will give necessary and sufficient conditions for the existence of Pythagorean triples in which all members are triangular. We will extend these conditions to discuss the problem of triangulars being represented as sums of powers.

2. PYTHAGOREAN TRIPLES WITH TRIANGULAR SOLUTIONS

By a triangular solution to a Diophantine equation $f(x, \ldots, x_n) = 0$, we mean a solution in which every variable is triangular.

Theorem 1: The Pythagorean equation $x^2 + y^2 = z^2$ has a triangular solution $x = T_a$, $y = T_b$, $z = T_c$ if and only if there exist integers m and k such that

$$T_b^2 = m^3 + (m+1)^3 + \cdots + (m+k)^3;$$

that is, T_b^2 is a sum of $k + 1$ consecutive cubes.