ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to
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Each solution (or problem) should be submitted on a separate sheet of paper.
Preference will be given to solutions (or problems) typed, double-spaced, in
the format used below. Solutions (or problems) should be received no later
than four months following (or prior to) the publication date.

DEFINITIONS
The Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) satisfy
\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1
\]
and
\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]
Also, \( \alpha \) and \( \beta \) designate the roots \((1 + \sqrt{5})/2 \) and \((1 - \sqrt{5})/2 \), respectively,
of \( x^2 - x - 1 = 0 \).

PROBLEMS PROPOSED IN THIS ISSUE

B-406 Proposed by Wray G. Brady, Slippery Rock State College, PA.

Let \( x_n = 4L_{3n} - L_n^3 \) and find the greatest common divisor of the terms of
the sequence \( x_1, x_2, x_3, \ldots \).

B-407 Proposed by Robert M. Giuli, Univ. of California, Santa Cruz, CA.

Given that
\[
\frac{1}{1 - x - xy} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} x^n y^k
\]
is a double ordinary generating function for \( a_{nk} \), determine \( a_{nk} \).

B-408 Proposed by Lawrence Somer, Washington, D.C.

Let \( d \in \{2, 3, \ldots \} \) and \( G_n = F_{dn}/F_n \). Let \( p \) be an odd prime and \( x = z(p) \)
be the least positive integer \( n \) with \( F_n \equiv 0 \pmod{p} \). For \( d = 2 \) and \( z(p) \) an
even integer \( 2k \), it was shown in B-386 that
\[
P_{n+1} G_{n+k} \equiv P_n G_{n+k+1} \pmod{p}.
\]
Establish a generalization for \( d \geq 2 \).

B-409 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.

Let \( P_n = P_n F_{K+a} \).

Must \( P_{n+5r} - P_n \) be an integral multiple of \( P_{n+4r} - P_{n+2r} \) for all nonnegative integers \( a \) and \( r \)?
B-410 Proposed by M. Wachtel, Zürich, Switzerland.

Some of the solutions of
\[ 5(x^2 + x) + 2 = y^2 + y \]
in positive integers \( x \) and \( y \) are:
\[ (x,y) = (0,1), (1,3), (10,23), (27,61). \]

Find a recurrence formula for the \( x_n \) and \( y_n \) of a sequence of solutions \( (x_n, y_n) \). Also find \( \lim(x_{n+1}/x_n) \) and \( \lim(x_{n+2}/x_n) \) as \( n \to \infty \) in term of \( a = (1 + \sqrt{5})/2 \).

B-411 Proposed by Bart Rice, Crofton, MD.

Tridiagonal \( n \) by \( n \) matrices \( A_n = (a_{ij}) \) of the form
\[ a_{ij} = \begin{cases} 2a \ (a \text{ real}) & \text{for } j = i \\ 1 & \text{for } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases} \]

occur in numerical analysis. Let \( d_n = \det A_n \).

(i) Show that \( \{d_n\} \) satisfies a second-order homogeneous linear recursion.

(ii) Find closed-form and asymptotic expressions for \( d_n \).

(iii) Derive the combinatorial identity
\[ \sum_{k=0}^{(n-1)/2} \binom{n}{2k+1} (-1)^k = (x+1)^{(n-1)/2} \frac{\sin \pi^2}{\sin \pi}, \]
\[ x > 0, \ t = \tan^{-1}(x). \]

SOLUTIONS

Lucky \( L \) Units Digit

B-382 Proposed by A.G. Shannon, N.S.W. Inst. of Technology, Australia.

Prove that \( L_n \) has the same last digit (i.e., units digit) for all \( n \) in the infinite geometric progression 4, 8, 16, 32, ...

Note: Several solvers pointed out that the subscript \( n \) was missing from the \( L_n \).

Solution by Lawrence Somer, Washington, D.C.

I present two solutions, the first of which is more direct.

**First Solution:** Note that
\[ L_n^2 = (a^n + b^n)^2 = a^{2n} + b^{2n} + 2(ab)^n = L_{2n} + 2(-1)^n. \]

We now proceed by induction. \( L_4 = 7 \). Now assume
\[ L_{2n} \equiv 7 \pmod{10}, \ n \geq 2. \]

Then
\[ L_{2n+1} + 2(-1)^n = L_{2n+1} + 2 = L_{4n}^2 \equiv 7^2 \equiv 9 \pmod{10}. \]
Thus,
\[ L_{2n+1} \equiv 9 - 2 \equiv 7 \pmod{10} \]
and we are done.

**Second Solution:** Note that the Lucas sequence has a period modulo 10 of 12.
Now \( (2^n) \pmod{12} \) is of the form 4, 8, 4, 8, \ldots. But \( L_n \) and \( L_B \) both end in 7. Thus, we are done.


### Reappearance

**B-383** Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA.

Solve the difference equation
\[ U_{n+2} - 5U_{n+1} + 6U_n = F_n. \]

**Note:** Bob Prielipp and Sahib Singh point out that B-383 is a rerun of B-370.

Solvers in addition to those of B-370 are Ralph Garfield, Lawrence Somer, and Gregory Wulczyn.

A Recursion for \( F_{2m}^b \) or \( F_{2m+1}^b \)

**B-384** Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA.

Establish the identity
\[ F_{n+10}^b = 55(F_{n+8}^b - F_{n+2}^b) - 385(F_{n+6}^b - F_{n+4}^b) + F_n^b. \]

Solution by Sahib Singh, Clarion State College, Clarion, PA.

It suffices to prove that
\[ F_{n+10}^b - F_n^b = 55(F_{n+8}^b - F_{n+2}^b) - 385(F_{n+6}^b - F_{n+4}^b). \]

Factoring the difference of squares, one sees that (1) follows from the two formulas:
\[ \frac{F_{n+10}^b - F_n^b}{55} = \frac{(F_{n+8}^b - F_{n+2}^b)}{55} = \frac{F_{n+8}^b - F_{n+2}^b}{55}; \]
\[ \frac{F_{n+10}^2 + F_n^2}{55} = 8(F_{n+8}^2 + F_{n+2}^2) - 7(F_{n+6}^2 + F_{n+4}^2). \]

Each of (2) and (3) can be established using the Binet formulas, \( a^2 = a + 1 \), \( a^3 = 3a + 2 \), etc., and the corresponding formulas for powers of \( b \).

Also solved by Paul S. Bruckman and the proposer.

### Counting Some Triangular Numbers

**B-385** Proposed by Herta T. Freitag, Roanoke, VA.

Let \( T_n = n(n+1) / 2 \). For how many positive integers \( n \) does one have both \( 10^6 < T_n < 2 \cdot 10^6 \) and \( T_n \equiv 8 \pmod{10} \)?

Solution by Lawrence Somer, Washington, D.C.

By inspection, \( T_n \equiv 8 \pmod{10} \) if and only if
\[ n \equiv 7 \pmod{20} \text{ or } n \equiv 12 \pmod{20}. \]
Now, \(10^6 < T < 2 \cdot 10^6\) if and only if
\[
-1/2 + \sqrt{2,000,000.25} < n < -1/2 + \sqrt{4,000,000.25}
\]
or \(1414 < n < 1999\). There are 58 integers satisfying conditions (1) and (2). The answer is thus 58.

Also solved by Paul S. Bruckman, Sahib Singh, Charles W. Trigg, Gregory Wulczyn, and the proposer.

**Elusive Generalization**

B-386 Proposed by Lawrence Somer, Washington, D.C.

Let \(p\) be a prime and let the least positive integer \(m\) with \(F_m \equiv 0 \pmod{p}\) be an even integer \(2k\). Prove that
\[
F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p}.
\]

Generalize to other sequences, if possible.

Solution by Paul S. Bruckman, Concord, CA.

The following formula may be readily verified from the Binet definitions:
\[
F_{n+1}L_{n+k} = F_nL_{n+k+1} = (-1)^k L_k.
\]

Since \(F_{2k} = F_kL_k \equiv 0 \pmod{p}\) and \(2k\) is the least positive integer \(m\) such that \(F_m \equiv 0 \pmod{p}\), thus \(F_k \not\equiv 0 \pmod{p}\), which implies \(L_k \equiv 0 \pmod{p}\). From (1), we see that this, in turn, implies
\[
F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p}.
\]

The desired generalization to other sequences appears to be elusive.

**Editor's note:** For one generalization, see B-408, proposed in this issue.

Also solved by Sahib Singh, Gregory Wulczyn, and the proposer.

**One's Own Infinitude**

B-387 Proposed by George Berzsenyi, Lamar Univ., Beaumont, TX.

Prove that there are infinitely many ordered triples of positive integers \((x, y, z)\) such that
\[
3x^2 - y^2 - z^2 = 1.
\]

**Editor's note:** An infinite number of solutions were produced with \(y = z + 2\) by Paul S. Bruckman, with \(y = z\) and with \(z = 1\) by Bob Prielipp, with \(x = z\) by Sahib Singh, with \(x \equiv 1\) by Gregory Wulczyn, and with \((x^2, y^2, z^2) = (F_{2n+2}, F_{2n}, x^2)\) by Gregory Wulczyn, where \(x = 4F_{2n+1}F_{2n+2}F_{2n+3}F_{2n+4}\), by the proposer. Of these, the following was chosen for publication because of its bibliographic and historical references.

Solution by Bob Prielipp, Univ. Of Wisconsin-Oshkosh, WI.

We will show that there are infinitely many ordered triples of positive integers \((x, y, 1)\) such that
\[
3x^2 - y^2 - 1^2 = 1.
\]

The preceding equation is equivalent to \(y^2 - 3x^2 = -2\). To assist us in finding all of its solutions, we will employ the following results:
(1) If \( D \) and \( N \) are positive integers, and if \( D \) is not a perfect square, the equation \( u^2 - Dv^2 = -N \) has a finite number of classes of solutions. If \((u^*,v^*)\) is the fundamental solution of the class \( K \), we obtain all the solutions \((u,v)\) of \( K \) by the formula
\[ u + v\sqrt{D} = (u^* + v^*\sqrt{D})(a + t\sqrt{D}), \]
where \((a,t)\) runs through all solutions of \( a^2 - Dt^2 = 1 \), including \((\pm 1,0)\).

(2) If \( p \) is a prime, and if the equation \( u^2 - Dv^2 = -p \) is solvable, it has one or two classes of solutions, according as the prime \( p \) divides \( 2D \) or not. [See Nagell, *Introduction to Number Theory* (2nd ed.; New York: Chelsea Publishing Company, 1964), pp. 204-208.]

By (2), there is only one class of solutions of the equation
\[ y^2 - 3x^2 = -2. \]
The fundamental solution is \((1,1)\). The fundamental solution of the equation \( y^2 - 3x^2 = 1 \) is \((2,1)\). So, all positive integer solutions of \( y^2 - 3x^2 = -2 \) are given by the formula
\[ y + x\sqrt{3} = (1 + \sqrt{3})(2 + \sqrt{3})^n, \quad n = 0, 1, 2, 3, \ldots. \]
Thus, the first six positive integer solutions \((y,x)\) of \( y^2 - 3x^2 = -2 \) are
\[ (1,1), (5,3), (19,11), (71,41), (265,153), \text{ and } (989,571). \]
The corresponding six positive integer solutions \((x,y,z)\), with \( z = 1 \), of the equation \( 3x^2 - y^2 - z^2 = 1 \) are
\[ (1,1,1), (3,5,1), (11,19,1), (41,71,1), (153,165,1), \text{ and } (571,989,1). \]

It may be of interest to note that, in his famous *Measurement of a Circle*, Archimedes determines that \( \frac{22}{7} > \pi > \frac{10}{3} \) and in deducing these inequalities he uses \( \frac{1351}{780} > \sqrt{3} > \frac{265}{153} \). It can be shown that these good approximations to \( \sqrt{3} \) satisfy the equations \( a^2 - 3b^2 = 1 \) and \( a^2 - 3b^2 = -2 \), respectively, so that Archimedes knew at least some solutions of these equations.