A CONJECTURE IN GAME THEORY

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We consider a team composed of \(n\) players, with each member playing the same \(r\) games, \(G_1, G_2, \ldots, G_r\). We assume that each game \(G_j\) has two possible outcomes, success and failure, and that the probability of success in game \(G_j\) is equal to \(p_j\) for each player. We let \(X_{ij}\) be equal to one (1) if player \(i\) has a success in game \(j\) and let \(X_{ij}\) be equal to zero (0) if player \(i\) has a failure in game \(j\). We assume throughout this paper that the random variables \(X_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, r\) are independent.

Let \(S_{jn}\) denote the total number of successes in the \(j\)th game. We define the point-value of a team to be

\[
\Psi_n = \min_{1 \leq j \leq r} S_{jn}.
\]

This means that the point-value of a team is equal to the minimum number of successes in any particular game. Clearly,

\[
P(S_{jn} = m) = \binom{n}{m} p_j^m (1 - p_j)^{n-m}, \quad m = 0, 1, 2, \ldots, n,
\]

and

\[
E[\Psi_n] = \sum_{k=0}^{n} k \cdot P(\Psi_n = k) = \sum_{k=0}^{n-1} P(\Psi_n > k)
\]

\[
= \sum_{k=0}^{n-1} P(S_1 > k, S_2 > k, \ldots, S_r > k)
\]

\[
= \sum_{k=0}^{n-1} \prod_{j=1}^{r} P(S_{jn} > k)
\]

\[
= \sum_{k=0}^{n-1} \prod_{j=1}^{r} \sum_{m=k+1}^{n} \binom{n}{m} p_j^m (1 - p_j)^{n-m}.
\]

It follows from the definition of \(\Psi_n\) that the expected point-value for a team is an increasing function of \(n\), i.e.,

\[E[\Psi_n] \leq E[\Psi_{n+1}], \quad n = 1, 2, 3, \ldots\]

Since a team can add players in order to increase its expected point-value, it seems reasonable to define the score to be the expected point-value per player. Namely, we denote the score by

\[
W_n = \frac{1}{n} E[\Psi_n].
\]
Thus, from (1), we obtain

\[(2) \quad W_n = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{n} \sum_{m=0}^{n} \binom{n}{m} p_j^n (1 - p_j)^{n-m}.\]

It is not obvious from (2) how the score varies as the number of players increases. We now prove that \( W_n \) is a strictly increasing function of \( n \) in the special case \( r = 2 \) and \( p_1 = p_2 \). We first prove three lemmas, which are also of independent interest.

**Lemma 1:** Let a team be composed of \( j \) players, with each member playing the same two games, \( G_1 \) and \( G_2 \). Let the probability of success for each player in both games \( G_1 \) and \( G_2 \) be equal and be denoted by \( p \). Let \( u_j = P\{S_1j = S_2j\} \), for all positive integers \( j \). Then

\[\frac{1}{2\Pi} \int_0^{2\Pi} |p + qe^{i\theta}|^2 \, d\theta = u_j,\]

where \( q = 1 - p \).

**Proof:** Using the fact that

\[P\{S_1j = m\} = \binom{j}{m} p^m (1 - p)^{j-m}, \quad m = 0, 1, 2, \ldots, j, \quad j, \, i = 1, 2,\]

and the independence of the random variables \( S_1j \) and \( S_2j \), we obtain

\[(3) \quad u_j = \sum_{m=0}^{j} \binom{j}{m}^2 p^{2m} (1 - p)^{2(j-m)}, \quad j = 1, 2, 3, \ldots.\]

We note that if \( f \) is the polynomial \( f(n) = \sum_{n=0}^{j} a_n n^n \), then

\[(4) \quad \frac{1}{2\Pi} \int_0^{2\Pi} |f(e^{i\theta})|^2 \, d\theta = \sum_{n=0}^{j} a_n^2.\]

We now apply the binomial expansion and (4) to the function \( f(n) = (p + qz)^j \), where \( j \) is a positive integer. The binomial expansion yields

\[f(n) = (p + qz)^j = \sum_{n=0}^{j} \binom{j}{n} p^n q^{j-n} z^{j-n},\]

and using (3) and (4), we obtain

\[(5) \quad \frac{1}{2\Pi} \int_0^{2\Pi} |p + qe^{i\theta}|^2 \, d\theta = \sum_{n=0}^{j} \binom{j}{n}^2 p^{2n} q^{2(j-n)} = u_j.\]

**Lemma 2:** Let \( r = 2, \, p_1 = p_2, \) and \( u_j = P\{S_1j = S_2j\} \), for all positive integers \( j \). Then \( u_j < u_{j-1} \).

**Proof:** Since

\[|p + qe^{i\theta}|^2 \leq 1, \quad 0 \leq \theta \leq 2\Pi\]

and

\[|p + qe^{i\theta}|^2 < 1, \quad 0 < \theta < 2\Pi,\]

the desired result follows from (5).
Lemma 3: Let \( u_j = P\{S_{1j} = S_{2j}\} \), for all positive integers \( j \) and let \( u_0 = 1 \). Let \( d_j = \Psi_{j+1} - \Psi_j \), \( j = 0, 1, 2, \ldots \), and let \( \Psi_0 = 0 \). Then

\[
\]

Proof: Clearly, \( d_j \) can assume only the values 0 and 1 with the following probabilities:

\[
P(d_j = 0) = 1 - [u_j p^2 + (1 - u_j)p],
\]

\[
P(d_j = 1) = u_j p^2 + (1 - u_j)p.
\]

Since \( E[d_j] = 0 \cdot P(d_j = 0) + 1 \cdot P(d_j = 1) \), we obtain the desired result.

Theorem: Let a team be composed of \( n \) players, with each member playing the same two games, \( G_1 \) and \( G_2 \). Let the probability of success for each player in both games \( G_1 \) and \( G_2 \) be equal and be denoted by \( p \). Then

\[
W_n < W_{n+1}, \ n = 1, 2, 3, \ldots
\]

Proof: Using the definition of \( W_n \), we obtain

\[
W_{n+1} - W_n = \frac{1}{n(n+1)} E[\Psi_{n+1} - \Psi_n] = \frac{1}{n(n+1)} E[\Psi_{n+1} - \Psi_n] - \frac{1}{n(n+1)} E[\Psi_n - \Psi_n].
\]

Using \( d_j \), as defined in Lemma 3, and noting that \( \Psi_n = \sum_{j=0}^{n-1} d_j \), (7) reduces to

\[
W_{n+1} - W_n = \frac{1}{n(n+1)} E\left[n(\Psi_{n+1} - \Psi_n) - \frac{n-1}{n(n+1)} \sum_{j=0}^{n-1} d_j\right].
\]

Using (6), we obtain

\[
W_{n+1} - W_n = \frac{1}{n(n+1)} \left[n(u_n p^2 + (1 - u_n)p) - \sum_{j=0}^{n-1} (u_j p^2 + (1 - u_j)p)\right].
\]

Thus, to prove that \( W_n < W_{n+1} \), it suffices to show that

\[
n(u_n p^2 + (1 - u_n)p) - \sum_{j=0}^{n-1} (u_j p^2 + (1 - u_j)p) > 0.
\]

Proving inequality (8) is equivalent to showing that

\[
nu_n - \sum_{j=0}^{n-1} u_j = \sum_{j=1}^{n} j(u_j - u_{j-1}) < 0.
\]

Since (9) follows from Lemma 2, we conclude that

\[
W_n < W_{n+1}, \ n = 1, 2, 3, \ldots
\]

It is the author's conjecture that in the general case discussed in the beginning of this paper \( (n > 2 \) and \( p_1 \) not necessarily equal to \( p_2 \) ) that \( W_n \) is a strictly increasing function of \( n \), too. The above proven theorem and some elementary numerical computations suggest the truth of this statement, but the author has not been able to supply a complete proof.

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