ABSORPTION SEQUENCES

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1. INTRODUCTION

In the classical gambler's ruin problem, a gambler beginning with i dollars, either wins or loses one dollar each play. The game ends when he has lost all his initial money or has accumulated $a(\geq i)$ dollars. The situation can also be described as a simple random walk on the integers beginning at with absorbing barriers at 0 and a. Let $F_a(i,n)$ represent the number of different paths of exactly n steps which begin at i ($i = 0, 1, 2, \ldots, a$) and end with absorption at either 0 or a. For fixed values of a and i, $F_a(i,n)$ is a sequence of nonnegative integers called an "absorption sequence." In other words, $F_a(i,n)$ represents the number of different ways a gambler who begins with i dollars can end his play using n one dollar bets.

2. A RECURRENCE RELATION WITH BOUNDARY CONDITIONS

Appropriate boundary conditions, suggested by the condition that the random walk stops when it first hits either 0 or a are

$$F_a(0,0) = F_a(a,0) = 1$$

$$F_a(i,0) = 0, \; i = 1, 2, \ldots, a - 1$$

$$F_a(0,n) = F_a(a,n) = 0, \; n \quad 0.$$

A path which begins at $0 < i < a$ must in one step go to either $i - 1$ or $i + 1$. For this reason, we have a recurrence relation for the number of paths:

$$F_a(i,n) = F_a(i - 1,n - 1) + F_a(i + 1,n - 1), \; n > 0, \; 0 < i < a.$$

3. EXAMPLES OF RECURRENCE RELATIONS AND ABSORPTION SEQUENCES

TABLE 1. $F_5(i,n)$

i \ n	0	1	2	3	4	5	6	7	8	9	10	11	12
5	1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	1	1	2	3	5	8	13	21	34	55
3	0	0	1	1	2	3	5	8	13	21	34	55	89
2	0	0	1	1	2	3	5	8	13	21	34	55	89
1	0	1	0	1	1	2	3	5	8	13	21	34	55
0	1	0	0	0	0	0	0	0	0	0	0	0	0

The entries in each row are the beginning of an absorption sequence. Absorption at 0 or 5.

TABLE 2. $F_9(i,n)$

i \ n	0	1	2	3	4	5	6	7	8	9	10	11	12
9	1	0	0	0	0	0	0	0	0	0	0	0	0
8	0	1	0	1	0	2	0	5	1	14	7	42	34
7	0	0	1	0	2	0	5	1	14	7	42	34	132
6	0	0	0	1	0	3	1	9	6	28	27	90	109
5	0	0	0	0	1	1	4	5	14	20	48	75	165
4	0	0	0	0	1	1	4	5	14	20	48	75	165
3	0	0	0	1	0	3	1	9	6	28	27	90	109
2	0	0	1	0	2	0	5	1	14	7	42	34	132
1	0	1	0	1	0	2	0	5	1	14	7	42	34
0	1	0	0	0	0	0	0	0	0	0	0	0	0

The entries in eqch row are the beginning of an absorption sequence. Absorption at 0 or 9.

(a) $F_3(1,n) = F_3(2,n) = 1,\ n > 0.$

(b) $F_4(1,2m) = 0,\ F_4(1,2m+1) = 2^m,\ m \geq 0;$

$F_4(2,2m) = 2^m,\ m > 0,\ F_4(2,2m+1) = 0,\ m \geq 0.$

(c) Let F_n represent the well-known Fibonacci number sequence [1]:

$$F_1 = 1,\ F_2 = 1,\ \ldots,\ F_{n+1} = F_n + F_{n-1}$$

in general. We have

$$F_5(1,n+2) = F_5(2,n+1) = F_n \quad \text{(see Table 1)}$$
$$F_5(1,n) = F_5(4,n),\ F_5(2,n) = F_5(3,n)$$

by symmetry.

By enumerating, see Table 1, it is easy to show that (assuming $a = 5$ and omitting the subscript)

$$F(2,2) = F(2,3) = 1$$
$$\begin{aligned} F(2,n+1) &= F(1,n) + F(3,n) \quad \text{(recurrence relation)} \\ &= F(1,n) + F(2,n) \quad \text{(symmetry)} \\ &= F(2,n-1) + F(2,n) \quad \text{(boundary condition for } n > 1\text{)}. \end{aligned}$$

The sequence $F(2,n)$ thus satisfies the initial conditions and recurrence relation for the Fibonacci numbers. In the case of $F_3(1,n)$, the argument is similar.

(d) $F_6(1,2m) = 0,\ F_6(1,2m+1) = 3^{m-1},\ m \geq 1$, and $F_6(1,1) = 1;$

$F_6(2,2m) = 3^{m-1},\ m \geq 1,\ F_6(2,2m+1) = 0;$

$F_6(3,2m) = 0,\ F_6(3,2m+1) = 2 \cdot 3^{m-1},\ m \geq 1$, and $F_6(3,1) = 0.$

(e) Let $a = 9$ and omit the subscript.

$$F(1,1) = 1,\ F(1,2) = 0,\ F(1,3) = 1$$

and

$$F(1,n) = 3F(1,n-2) + F(1,n-3) - 1,\ n > 3.$$

$$F(2,1) = 0,\ F(2,2) = 1,\ F(2,3) = 0$$

and

$$F(2,n) = 3F(2,n-2) + F(2,n-3) - 1,\ n > 3.$$

$$F(3,1) = 0,\ F(3,2) = 0,\ F(3,3) = 1$$

and

$$F(3,n) = 3F(3,n-2) + F(3,n-3),\ n > 3.$$

$$F(4,1) = 0,\ F(4,2) = 0,\ F(4,3) = 0$$

and

$$F(4,n) = 3F(4,n-2) + F(4,n-3) + 1,\ n > 3.$$

$F(9 - i,n) = F(i,n)$ by symmetry.

By enumeration, see Table 2, the initial conditions can be seen to hold as well as the fact that (assuming $a = 9$ and omitting the subscript)

$$F(1,4) = 0,\ F(2,4) = 2,\ F(3,4) = 0, \text{ and } F(4,4) = 1.$$

The recurrence relations therefore hold if $n = 4$. For an induction argument assume they all hold for a general value of n.

$$\begin{aligned} F(1,n+1) &= F(0,n) + F(2,n) = F(2,n) \\ &= 3F(2,n-2) + F(2,n-3) - 1 \quad \text{(the induction hypothesis)} \\ &= 3F(1,n-1) + F(1,n-2) - 1 \end{aligned}$$

[for $i > 0$, $F(0,i) = 0$.] Similarly,

$$\begin{aligned} F(2,n+1) &= F(1,n) + F(3,n) \\ &= 3[F(1,n-2) + F(3,n-2)] + F(1,n-3) \\ &\qquad + F(3,n-3) - 1 \quad \text{(the induction hypothesis)} \\ &= 3F(2,n-1) + F(2,n-2) - 1. \end{aligned}$$

In just the same way, it is easy to show that both $F(3,n+1)$ and $F(4,n+1)$ satisfy, respectively, the stated recurrence relation.

(f) Assume $a = 10$ and omit the subscript.

$F(1,2m) = 0$, $F(1,1) = 1$, $F(1,3) = 1$, and

$$F(1,2m+1) = 4F(1,2m-1) - \sum_{k=1}^{m-1} F(1,2k-1) - 1,\ m > 1.$$

$F(2,2m-1) = 0$, $m \geq 1$, $F(2,2) = 1$, $F(2,4) = 2$, and

$$F(2,2m+2) = 4F(2,2m) - \sum_{k=1}^{m-1} F(2,2k) - 2.$$

$F(3,2m) = 0,\ m \geq 0,\ F(3,1) = 0,\ F(3,3) = 1$, and

$$F(3,2m+1) = 4F(3,2m-1) - \sum_{k=1}^{m-1} F(3,2k-1) - 1,\ m > 1.$$

$F(4,2m+1) = 0,\ m \geq 0,\ F(4,2) = 0,\ F(4,4) = 1$, and

$$F(4,2m+2) = 4F(4,2m) - \sum_{k=1}^{m-1} F(4,2k) + 1.$$

$F(5,2m) = 0,\ m \geq 0,\ F(5,1) = 0,\ F(5,3) = 0$, and

$$F(5,2m+1) = 4F(5,2m-1) - \sum_{k=1}^{m-1} F(5,2k-1) + 2.$$

$F_{10}(10 - i,n) = F_{10}(i,n),\ i = 1, 2, 3, 4$, by symmetry.

In the manner shown in example (e), all of these statements can be verified easily. Because of their length and repetitive nature, this discussion is omitted.

A referee has noted that if $A = (a_{ij})$ is the square matrix of order a defined by $a_{ij} = 1$ if $|i - j| = 1,\ i \neq 1,\ i \neq a;\ a_{ij} = 0$ otherwise, then the nth column X_n in the array of absorption sequences is given by

$$A^n X_0 = X_n \quad \text{where} \quad X_0 = (1, 0, 0, \ldots, 0, 1)^T.$$

This approach, as it has been applied to the related problem of counting paths in reflections in glass plates [2], might be used to codify and expand many of the current results. The referee has also made a (apparently correct) conjecture: if p is a prime and $a = 2p$, then p divides $F_{2p}(i,n)$ for $n \geq (p + 1)$ and $0 \leq 1 \leq 2p$.

4. RESULTS FOR SEQUENCES USING PROBABILISTIC REASONING

To illustrate what results follow from the connection between absorption sequences and probability, let us use the Fibonacci number sequence, F_n, which appears in example 3(c). Similar results can be found for any absorption sequence.

(a) The probability that absorption at one of the boundaries will take place is one [2, p. 345]. In the case where zero and five are the boundaries, $F_5(2,n)$ represents the number of paths that begin at two, and end at zero or five in n steps. If a "win" or a "loss" is equally likely, then the probability that the game is over in n steps is $2^{-n}F_5(2,n)$. Hence,

$$\sum_{n=1}^{\infty} 2^{-n}F_5(2,n) = 1 \quad \text{or} \quad \sum_{n=2}^{\infty} 2^{-n}F_{n-1} = 1.$$

(b) The expected duration of play in the equally likely case is given, in general, by the formula $i(a - i)$ [2, p. 349]. It is also given in this example by

$$\sum_{n=2}^{\infty} n2^{-n}F_5(2,n)$$

from the definition of expected value. We have then, with $a = 5$ and $i = 2$, that for the Fibonacci sequence

$$\sum_{n=2}^{\infty} n2^{-n}F_{n-1} = 6.$$

(c) In a formula attributed to Lagrange [2, p. 353] for the equally likely case with absorptions at 0 or 5, the probability of ruin (or absorption at zero) on the nth step is given as

$$u(i,n) = \frac{1}{5}\sum_{\nu=1}^{4}\left(\cos\frac{\pi\nu}{5}\right)^{n-1}\sin\frac{\pi\nu}{5}\sin\frac{\pi i\nu}{5}$$

$$i = 1,\ 2,\ 3,\ 4, \text{ and } n > 0.$$

In this formula, if $(n - i)$ is odd, $u(i,n) = 0$, as seems logical in terms of the random walk formulation as well as in light of trigonometric identities. If $(n - i)$ is even,

$$u(i,n) = \frac{2}{5}\left[\left(\cos\frac{\pi}{5}\right)^{n-1}\sin\frac{\pi}{5}\sin\frac{\pi i}{5} + \left(\cos\frac{2\pi}{5}\right)^{n-1}\sin\frac{2\pi}{5}\sin\frac{2\pi i}{5}\right].$$

Since, furthermore, each path of length n has probability 2^{-n}, the *number* of paths of length n involved is $2^n u(i,n)$. In particular, if $i = 3$, $n = 2m + 1$, then $2^{2m+1}u(3,2m + 1)$, which, as shown above, is the Fibonacci number F_{2m}. We obtain a trigonometric representation for "one-half" the Fibonacci numbers:

$$F_{2m} = \frac{2^{2m+2}}{5}\left[\left(\cos\frac{\pi}{5}\right)^{2m}\sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \left(\cos\frac{2\pi}{5}\right)^{2m}\sin\frac{2\pi}{5}\sin\frac{6\pi}{5}\right],$$

$$m = 1,\ 2,\ 3,\ \ldots\ .$$

To use Lagrange's probability of ruin formula for the rest of the Fibonacci numbers, the number of paths that begin at 2 and are absorbed at 0 in $2m$ steps for $m > 0$ is, as indicated above, $F(2,2m)$ or F_{2m-1}. Therefore, we have $2^{2m}u(2,2m) = F_{2m-1}$ or

$$F_{2m-1} = \frac{2^{2m+1}}{5}\left[\left(\cos\frac{\pi}{5}\right)^{2m-1}\sin\frac{\pi}{5}\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5}\right)^{2m-1}\sin\frac{2\pi}{5}\sin\frac{4\pi}{5}\right],$$

$$m = 1,\ 2,\ 3,\ \ldots\ .$$

Using trigonometric identities, these two formulas combine into one new trigonometric representation of the Fibonacci numbers.

$$F_n = \frac{2^{n+2}}{5}\ \left(\cos\frac{\pi}{5}\right)^n\sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \left(-\cos\frac{2\pi}{5}\right)^n\sin\frac{2\pi}{5}\sin\frac{6\pi}{5}\ ,\ n > 0.$$

(d) By using the method of images, repeatedly reflecting the path from the end points [2, p. 96], it is possible to show that in the random walk beginning at 3 with absorption at 0 or 5, the number of paths that arrive at 1 in $(n - 1)$ steps hitting neither 0 nor 5 is given by

$$\sum_k\left[\binom{n-1}{\frac{n+10k+1}{2}} - \binom{n-1}{\frac{n+10k+3}{2}}\right]$$

where the sum extends over the positive and negative integers k with the convention that the "binomial coefficient" $\binom{n}{x}$ is zero whenever x does not equal an integer between 0 and n. (This sum has a finite number of n n-zero terms.) With $n = 2m + 1$, it follows that the number of paths which are absorbed at 0 in $2m + 1$ steps is

$$F(3, 2m + 1) = F_{2m} = \sum_k \left[\binom{2m}{m + 5k + 1} - \binom{2m}{m + 5k + 2}\right].$$

To obtain the "other half" of the Fibonacci numbers, we count

$$F_{2m-1} = F(2, 2m),$$

the number of paths that begin at 2 and are absorbed at 0 in $2m$ steps. The method of repeated reflections gives us

$$F_{2m-1} = \sum_k \left[\binom{2m - 1}{m + 5k} - \binom{2m - 1}{m + 5k + 1}\right]$$

the sum extending over all positive and negative integers.

Two slightly different representations of the Fibonacci numbers can now be obtained through use of the easily verified relations

$$\binom{2m}{m + 5k + 1} - \binom{2m}{m + 5k + 2} = \frac{10k + 3}{2m + 1}\binom{2m + 1}{m + 5k + 2}$$

and

$$\binom{2m - 1}{m + 5k} - \binom{2m - 1}{m + 5k + 1} = \frac{5k + 1}{m}\binom{2m}{m + 5k + 1}$$

where k is any integer, m is a positive integer, and the conventions for the binomial coefficients introduced above continue to apply. By direct substitution, we obtain

$$F_{2m} = \sum_k \frac{10k + 3}{2m + 1}\binom{2m + 1}{m + 5k + 2} \quad \text{and} \quad F_{2m-1} = \sum_k \frac{5k + 1}{m}\binom{2m}{m + 5k + 1}.$$

Finally, by treating the terms with positive k separately from those with negative k, we obtain

$$F_{2m} = \frac{1}{2m + 1}\left\{\sum_{k=0}^{r}(10k + 3)\binom{2m + 1}{m + 5k + 2} - \sum_{k=1}^{s}(10k - 3)\binom{2m + 1}{m + 5k - 1}\right\},$$

$$F_{2m+1} = \frac{1}{m}\left\{\sum_{k=0}^{r}(5k + 1)\binom{2m}{m + 5k + 1} - \sum_{k=1}^{t}(5k - 1)\binom{2m}{m + 5k - 1}\right\},$$

$$r = \left[\frac{m - 1}{5}\right], \quad s = \left[\frac{m + 2}{5}\right], \quad t = \left[\frac{m + 1}{5}\right],$$

with [ ] the greatest integer in x, and the convention that a sum is zero if its lower limit exceeds its upper limit.

REFERENCES

1. R. A. Brualdi. *Introductory Combinatorics*. New York: North-Holland, 1977. Pp. 90-96.
2. W. Feller. *An Introduction to Probability Theory and Its Applications*. Vol. I; 3rd ed., rev. New York: Wiley, 1968.

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