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ON EULER'S SOLUTION OF A PROBLEM OF DIOPHANTUS

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1. The four numbers 1, 3, 8, 120 have the property that the product of any two of them is one less than a square. This fact was apparently discovered by Fermat. As one of the first applications of Baker's method in Diophantine approximations, Baker and Davenport [2] showed that there is no fifth positive integer n, so that

$$n + 1,\ 3n + 1,\ 8n + 1, \text{ and } 120n + 1$$

are all squares. It is not known how large a set of positive integers $\{x_1, x_2, \ldots, x_n\}$ can be found so that all $x_ix_j + 1$ are squares for all $1 \leq i < j \leq n$.

A solution attributed to Euler [1] shows that for every triple of integers x_1, x_2, y for which $x_1x_2 + 1 = y^2$ it is possible to find two further integers x_3, x_4 expressed as polynomials in x_1, x_2, y and a rational number x_5, expressed as a rational function in x_1, x_2, y; so that $x_ix_j + 1$ is the square of a rational expression x_1, x_2, y for all $1 \leq i < j \leq 5$.

In this note we analyze Euler's solution from a more abstract algebraic point of view. That is, we start from a field k of characteristic $\neq 2$ and adjoin independent transcendentals $x_1, x_2, \ldots, x_m$. We then set $x_ix_j + 1 = y_{ij}^2$ and pose two problems:

I. Find nonzero elements $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n$ in the ring $R = k[x_1, \ldots, x_m; y_{12}, \ldots, y_{m-1,m}]$ so that $x_ix_j + 1 = y_{ij}^2$; and $y_{ij} \in R$ for $1 \leq i < j \leq n$.

II. Find nonzero elements $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n$ in the field $K = k(x_1, \ldots, x_m; y_{12}, \ldots, y_{m-1,m})$ so that $x_ix_j + 1 = y_{ij}^2$; and $y_{ij} \in K$ for all $1 \leq i < j \leq n$.

In Section 2 we give a complete solution to Problem I for $m = 2$, $n = 3$. In Section 3 we give solutions for $m = 2$, $n = 4$ which include both Euler's

Research was supported in part by Grant MCS79-03162 from the National Science Foundation.

solution and a solution for $m = 3$, $n = 4$ which generalize the solutions mentioned above.

In Section 4 we present a solution for $m = 2$ or 3, $n = 5$ of Problem II, which again contains Euler's solution as a special case. Finally, in Section 5 we apply the results of Section 4 to Problem II for $m = 2$, $n = 3$.

The case char $k = 2$ leads to trivial solutions, $x = x_1 = x_2 = \cdots = x_n$, $y_{ij} = x + 1$.

Many of the ideas in this paper arose from conversations between Straus and John H. E. Cohn.

2. Solutions for $x_1x_3 + 1 = y_{13}^2$, $x_2x_3 + 1 = y_{23}^2$ with $x_3, y_{13}, y_{23} \in R = k[x_1, x_2, \sqrt{x_1x_2 + 1}]$.

We set $\sqrt{x_1x_2 + 1} = y_{12}$ and note that the simultaneous equations

$$(1)\qquad \begin{aligned} x_1x_3 + 1 &= y_{13}^2 \\ x_2x_3 + 1 &= y_{23}^2 \end{aligned}$$

lead to a Pell's equation

$$(2)\qquad x_1y_{23}^2 - x_2y_{13}^2 = x_1 - x_2.$$

In $R[\sqrt{x_1}, \sqrt{x_2}]$ we have the fundamental unit $y_{12} + \sqrt{x_1x_2}$ which, together with the trivial solution $y_{13} = y_{23} = 1$ of (2), leads to the infinite class of solutions of (2) which we can express as follows:

$$(3)\qquad y_{23}\sqrt{x_1} + y_{13}\sqrt{x_2} = \pm(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} + \sqrt{x_1x_2})^n;\ n = 0, \pm1, \pm2, \ldots .$$

In other words,

$$\pm y_{23}(n) = \frac{1}{2\sqrt{x_1}}[(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} + \sqrt{x_1x_2})^n + (\sqrt{x_1} \mp \sqrt{x_2})(y_{12} - \sqrt{x_1x_2})^n];$$

$$\pm y_{13}(n) = \frac{1}{2\sqrt{x_2}}[(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} + \sqrt{x_1x_2})^n - (\sqrt{x_1} \mp \sqrt{x_2})(y_{12} - \sqrt{x_1x_2})^n].$$

Once y_{13}, y_{23} are determined, then x_3 is determined by (1).

The cases $n = 1, 2$ give Euler's solutions:

$$y_{13}(1) = x_1 + y_{12},\ y_{23}(1) = x_2 + y_{12},\ x_3(1) = x_1 + x_2 + 2y_{12};$$

$$y_{13}(2) = 1 + 2x_1x_2 + 2x_1y_{12},\ y_{23}(2) = 1 + 2x_1x_2 + 2x_2y_{12};$$

$$x_3(2) = 4y_{12}[1 + 2x_1x_2 + (x_1 + x_2)y_{12}].$$

The interesting fact is that

$$x_3(1)x_3(2) + 1 = [3 + 4x_1x_2 + 2(x_1 + x_2)y_{12}]^2;$$

and in general

$$x_3(n)x_3(n + 1) + 1 = [x_3(n)y_{12} + y_{13}(n)y_{23}(n)]^2.$$

The main theorem of this section is the following (see [3] for a similar result).

Theorem 1: The general solution of (1) and (2) in R is given by (3).

We first need two lemmas.

Lemma 1: If $y_{13}, y_{23} \in R$ are solutions of (2), then, for a proper choice of the sign of y_{23}, we have

$$\eta = \frac{\sqrt{x_2}y_{13} - \sqrt{x_1}y_{23}}{\sqrt{x_2} - \sqrt{x_1}} \in R[\sqrt{x_1x_2}],$$

where η is a unit of $R[\sqrt{x_1x_2}]$.

Proof: Write $y_{13} = A + By_{12}$, $y_{23} = C + Dy_{12}$, where $A,B,C,D \in k[x_1,x_2]$. Then equation (2) yields

$$x_2 - x_1 = x_2(A + By_{12})^2 - x_1(C + Dy_{12})^2. \tag{4}$$

Under the homomorphism of R which maps $x_1 \to x$, $x_2 \to x$, we get

$$y_{12} \to \sqrt{x^2 + 1}A(x_1,x_2) \to A(x,x) = A(x), \text{ etc.},$$

and (4) becomes

$$0 = x[(A + C) + (B + D)y_{12}][(A - C) + (B - D)y_{12}]. \tag{5}$$

Thus, one of the factors on the right vanishes and by proper choice of sign, we may assume $A(x) = C(x)$, $B(x) = D(x)$, which is the same as saying that

$$\frac{A(x_1,x_2) - C(x_1,x_2)}{x_2 - x_1} = P, \quad \frac{B(x_1,x_2) - D(x_1,x_2)}{x_2 - x_1} = Q,$$

with $P,Q \in k[x_1,x_2]$. Thus,

$$\eta = \frac{\sqrt{x_2}y_{13} - \sqrt{x_1}y_{23}}{\sqrt{x_2} - \sqrt{x_1}} = y_{13} + \sqrt{x_1}(\sqrt{x_2} + \sqrt{x_1})(P + Qy_{12})$$

$$= y_{13} + (x_1 + \sqrt{x_1x_2})(P + Qy_{12}) \in R[\sqrt{x_1x_2}]$$

and, if we set

$$\bar{\eta} = \frac{\sqrt{x_2}y_{13} + \sqrt{x_1}y_{23}}{\sqrt{x_2} + \sqrt{x_1}} = y_{13} + (x_1 - \sqrt{x_1x_2})(P + Qy_{12})$$

we get $\eta\bar{\eta} = 1$.

Lemma 2: All units η of $R[\sqrt{x_1x_2}]$ are of the form

$$\eta = \kappa(y_{12} + \sqrt{x_1x_2})^n; \ \kappa \in k^*; \ n = 0, \pm 1, \ldots .$$

Proof: Write $x_1x_2 = s$, $x_1 = x$, $x_2 = s/x$, $t = \sqrt{s + 1}$. Then,

$$R = k[x,s/x,\sqrt{s + 1}] \subset k[x,1/x,t] = R^*.$$

We now consider the units, η^*, of $R^*[\sqrt{s}]$ and show that they are of the form:

$$\eta^* = \kappa x \ (t + \sqrt{t^2 - 1})^n, \ \kappa \in k^*; \ m,n \in Z. \tag{6}$$

Write $\eta^* = A + B\sqrt{t^2 - 1}$, where A and B are polynomials in t with coefficients in $k[x,1/x]$ and proceed by induction on deg A as a polynomial in t.

If deg $A = 0$, then $B = 0$ and A is a unit of $k[x,1/x]$, that is, $\eta = \kappa x^m$, $\kappa \in k^*$, $m \in \mathbb{Z}$.

Now assume the lemma true for deg $A < n$ and write

$$A = a_nt^n + a_{n-1}t^{n-1} + \cdots, \ B = b_{n-1}t^{n-1} + b_{n-2}t^{n-2} + \cdots .$$

Since η^* is a unit, we get that

$$\eta^*\eta^{-*} = A^2 - (t^2 - 1)B^2$$

is a unit of $k[x,1/x]$. So, comparing coefficients of t^{2n} and t^{2n-1}, we get:

$$a_n^2 = b_{n-1}^2,\ a_n a_{n-1} = b_{n-1} b_{n-2}$$

or

$$a_n = \pm b_{n-1},\ a_{n-1} = \pm b_{n-2}$$

Thus,

$$\eta^{**} = \eta^*(t \mp \sqrt{t^2 - 1}) = [tA \mp (t^2)B] + (tB \pm A)\sqrt{t^2 = 1}$$
$$= A_1 + B_1\sqrt{t^2 - 1},$$

where $A_1 = a_n t^{n+1} + a_{n-1}t^n + \cdots \mp (t^2 - 1)(a_n t^{n-1} + a_{n-1}t^{n-2} \ldots)$, so that deg $A_1 < n$ and η^{**} is of the form (6) by the induction hypothesis. Therefore $\eta^* = \eta^{**}(t \pm \sqrt{t^2 - 1})$ is also of the form (6).

Now η^* is a unit of $R[\sqrt{t^2 - 1}]$ if and only if κx^m is a unit of R; that is, if and only if $m = 0$.

Theorem 1 now follows directly from Lemmas 1 and 2 if we write

$$\sqrt{x_2}y_{13} + \sqrt{x_1}y_{23} = \kappa(\sqrt{x_2} \pm \sqrt{x_1})(y_{12} + \sqrt{x_1x_2})^n$$

and get

$$x_2y_{13}^2 - x_1y_{23}^2 = \kappa^2 = 1.$$

so that $\kappa = \pm 1$.

Note that Theorem 1 does not show that, for any two integers x_1, x_2 for which $x_1x_2 + 1$ is a square, all integers x_3 for which $x_ix_3 + 1$ are squares; $i = 1, 2$; are of the given forms. But these forms are the only ones that can be expressed as polynomials in x_1, x_2, $\sqrt{x_1x_2 + 1}$ and work for all such triples.

As mentioned above, we have the recursion relations

$$y_{13}(n + 1) = x_1y_{23}(n) + y_{12}y_{13}(n),$$
$$y_{23}(n + 1) = x_2y_{13}(n) + y_{12}y_{23}(n),$$
$$x_3(n + 1) = x_1 + x_2 + x_3(n) + 2x_1x_2x_3(n) + 2y_{12}y_{13}(n)y_{23}(n),$$

and therefore

$$x_3(n)x_3(n + 1) + 1 = [y_{12}x_3(n) + y_{13}(n)y_{23}(n)]^2, \tag{7}$$

so that the quadruple $x_1, x_2, x_3(n) = x_3, x_3(n + 1) + x_4$ has the property that $x_ix_j + 1$ is a square for $1 \le i < j \le 4$.

From [3, Theorem 3], we get the following.

Theorem 2: $x_3(m)x_3(n) + 1$ is a square in R if any only if $|m - n| = 1$.

Note that while the proof in [3] is restricted to a more limited class of solutions, the solutions there are obtained by specialization from the solutions presented here.

3. Solutions for $x_ix_4 + 1 = y_{i4}^2$; $i = 1, 2, 3$ with $x_4, y_{i4} \in R = k[x_1, x_2, x_3, y_{12}, y_{13}, y_{23}]$ where $y_{ij} = \sqrt{x_ix_j + 1}$; $1 \le i < j \le 3$.

The solution (7) using $x_3 = x_3(n)$, $x_4 = x_4(n)$ as polynomials in x_1, x_2, y_{12} can be generalized as follows.

Theorem 3: For $x_4 = x_1 + x_2 + x_3 + 2x_1x_2x_3 + 2y_{12}y_{13}y_{23}$, we have

$$x_ix_4 + 1 = y_{i4}^2,\ y_{i4} = x_iy_{jk} + y_{ij}y_{ik};\ \{i,j,k\} = \{1,2,3\}.$$

Proof: We have

$$y_{i4}^2 - 1 = -1 + x_i^2(x_jx_k + 1) + (x_ix_j + 1)(x_ix_k + 1) + 2x_iy_{12}y_{13}y_{23}$$
$$= x_i(x_1 + x_2 + x_3 + 2x_1x_2x_3 + 2y_{12}y_{13}y_{23})$$
$$= x_ix_4.$$

Note that since the choice of the sign of y_{ij} is arbitrary, we always get two conjugate solutions for $x_4 \in R$. This corresponds to the choices

$$x_4 = x_3(n \pm 1)$$

in the previous section.

Theorem 4: The values x_4 in Theorem 3 are the only nonzero elements of R with $x_i x_4 + 1$ squares in R for $i = 1, 2, 3$.

Proof: Let $x_4 = P(x_1, x_2, x_3, y_{12}, y_{13}, y_{23}) \in R$ where, in order to normalize the expression we assume that P is linear in the y_{ij} and $P \neq 0$. By Theorem 2, we have

$$P[x_1, x_2, x_3(n), y_{12}, y_{13}(n), y_{23}(n)] = x_3(n + 1)$$

for each $n = 0, \pm 1, \pm 2, \ldots$. Without loss of generality we may assume that $P = x_3(n + 1)$ for infinitely many choices of n. Then the algebraic function of x_3

$$P(x_1, x_2, x_3, y_{12}, y_{13}, y_{23}) - x_1 - x_2 - x_3 - 2x_1x_2x_3 - 2y_{12}y_{13}y_{23}$$

has infinitely many zeros $x_3 = x_3(n)$ and hence is identically 0.

The values x_4 in Theorem 3 can be characterized in the following symmetric way.

Lemma 3: Let σ_i; $i = 1, 2, 3, 4$ be the elementary symmetric functions of x_1, x_2, x_3, x_4. Then x_4 is the value given by Theorem 3 if and only if

$$\sigma_1^2 = 4(\sigma_2 + \sigma_4 + 1). \tag{8}$$

Proof: If we write Σ_1, Σ_2, Σ_3 for the elementary symmetric functions of x_1, x_2, x_3, then $x_4 = \Sigma_1 + 2\Sigma_3 + 2Y$ where

$$Y = y_{12}y_{13}y_{23} = \sqrt{\Sigma_3^2 + \Sigma_1\Sigma_3 + \Sigma_2 + 1}.$$

Hence

$$\begin{aligned} \sigma_1 &= 2(\Sigma_1 + \Sigma_3 + Y) \\ \sigma_2 &= \Sigma_2 + x_4\Sigma_1 = \Sigma_2 + \Sigma_1^2 + 2\Sigma_1\Sigma_3 + 2\Sigma_1 Y \\ \sigma_4 &= x_4\Sigma_3 = \Sigma_1\Sigma_3 + 2\Sigma_3^2 + 2\Sigma_3 Y. \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} \sigma_1^2 &= 4[\Sigma_1^2 + 2\Sigma_1\Sigma_3 + \Sigma_3^2 + 2\Sigma_1 Y + 2\Sigma_3 Y + Y^2] \\ &= 4[\sigma_2 + \sigma_4 - \Sigma_2 - \Sigma_1\Sigma_3 - \Sigma_3^2 + (x_1x_2 + 1)(x_1x_3 + 1)(x_2x_3 + 1)] \\ &= 4(\sigma_2 + \sigma_4 + 1). \end{aligned}$$

Conversely, if we solve the quadratic equation (8) for x_4, we get the two values in Theorem 3.

4. Solutions for $x_i x_5 = y_{i5}^2$; $i = 1, 2, 3, 4$ with $x_4, y_{i5} \in K = k(x_1, x_2, x_3, y_{12}, y_{13}, y_{23})$ where x_4 is given by Theorem 3.

If we use the x_4 of the previous section and define

$$x_5 = \frac{4\sigma_3 + 2\sigma_1 + 2\sigma_1\sigma_4}{(\sigma_4 - 1)^2} \tag{10}$$

we get the following.

Theorem 5: We have

$$x_i x_5 + 1 = \left(\frac{2x_i^2 - \sigma_1 x_i - \sigma_4 - 1}{\sigma_4 - 1}\right)^2; \; i = 1, 2, 3, 4.$$

Proof: The x_i are the roots of the equation

(11) $$x_i^4 - \sigma_1 x_i^3 + \sigma_2 x_i^2 - \sigma_3 x_i + \sigma_4 = 0.$$

Hence

(12) $$(\sigma_4 - 1)^2(x_i x_5 + 1) = 4\sigma_3 x_i + 2\sigma_1 x_i + 2\sigma_1\sigma_4 x_i + (\sigma_4 - 1)^2.$$

If we substitute $4\sigma_3 x_i = 4(x_i^4 - \sigma_1 x_i^3 + \sigma_2 x_i^2 + \sigma_4)$ from (11), we get

(13) $$\begin{aligned}(\sigma_4 - 1)^2(x_i x_5 + 1) &= 4x_i^4 - 4\sigma_1 x_i^3 + 4\sigma_2 x_i^2 + 2\sigma_1(\sigma_4 + 1)x_i + (\sigma_4 + 1)^2 \\ &= (2x_i^2 - \sigma_1 x_i - \sigma_4 - 1)^2 - (\sigma_1^2 - 4\sigma_4 - 4 - 4\sigma_2)x_i^2 \\ &= (2x_i^2 - \sigma_1 x_i - \sigma_4 - 1)^2,\end{aligned}$$

since the last bracket vanishes by Lemma 3.

Thus, the famous quadruple 1, 3, 8, 120 can be augmented by

$$x_5 = \frac{777480}{2879^2}.$$

We conjecture that the quintuple given by Theorem 5 is the only pair of quintuples in which x_4 is a polynomial in $x_1, x_2, x_3; y_{12}, y_{13}, y_{23}$ and x_5 is rational in these quantities.

Finally, we show that the value x_5 given by Theorem 5 is never an integer when $x_1, x_2, x_3, y_{12}, y_{13}, y_{23}$ and, hence, x_4 and y_{14}, y_{24}, y_{34} are positive integers.

Theorem 6: If the quantities $x_1, x_2, x_3, y_{12}, y_{13}, y_{23}$ in Theorem 5 are positive integers, then $0 < x_5 < 1$.

Proof: Since we have already verified the theorem for the case $x_1 = 1$, $x_2 = 3$, $x_3 = 8$, we may assume that

$$\frac{\Sigma_1}{\Sigma_3} = \frac{1}{x_1x_2} + \frac{1}{x_1x_3} + \frac{1}{x_2x_3} < \frac{1}{3} + \frac{1}{8} + \frac{1}{24} = \frac{1}{2},$$

and the smallest Σ_1 is obtained for the triple 2, 4, 12. Thus,

(14) $$18 \leq \Sigma_1 < \frac{1}{2}\Sigma_3.$$

Similarly

$$\frac{\Sigma_2}{\Sigma_3} < 1 + \frac{1}{3} + \frac{1}{8} < \frac{3}{2}$$

and

(15) $$80 \leq \Sigma_2 < \frac{3}{2}\Sigma_3.$$

Next, $Y = y_{12}y_{13}y_{23}$ satisfies $Y = \sqrt{\Sigma_3^2 + \Sigma_1\Sigma_3 + \Sigma_2 + 1}$, so that from (14) and (15) we get

(16) $$\Sigma_3 + 9 \leq Y < \frac{3}{2}(\Sigma_3 + 1).$$

Thus, the numerator of $1 - x_5$ is

(17) $$\begin{aligned}(\sigma_4 - 1)^2 - 2\sigma_1\sigma_4 - 4\sigma_3 - 2\sigma_1 &= (\sigma_4 - \sigma_1 - 1)^2 - \sigma_1^2 - 4\sigma_3 - 4\sigma_1 \\ &= (\sigma_4 - \sigma_1 - 1)^2 - 4(\sigma_4 + \sigma_2 + 1) \\ &\qquad - 4\sigma_3 - 4\sigma_1 \\ &= (\sigma_4 - \sigma_1 - 3)^2 - 4\sigma_3 - 4\sigma_2 - 8\sigma_1 - 8\end{aligned}$$

$$= (2\Sigma_3^2 + 2\Sigma_3 Y + \Sigma_1\Sigma_3 - 2\Sigma_3 - 2Y - 2\Sigma_1 - 3)^2 - 8\Sigma_2\Sigma_3 - 8\Sigma_2 Y$$
$$- 4\Sigma_1\Sigma_2 - 4\Sigma_3 - 8\Sigma_1\Sigma_3 - 8\Sigma_1 Y - 4\Sigma_1^2 - 4\Sigma_2 - 16\Sigma_3 - 16Y$$
$$- 16\Sigma_1 - 8$$
$$> (4\Sigma_3^2 + 30\Sigma_3 - 6)^2 - 12\Sigma_3^2 - 18\Sigma_3(\Sigma_3 + 1) - 3\Sigma_3^2 - 4\Sigma_3 - 4\Sigma_3^2$$
$$- 6\Sigma_3(\Sigma_3 + 1) - \Sigma_3^2 - 6\Sigma_3 - 16\Sigma_3 - 24(\Sigma_3 + 1) - 8\Sigma_3 - 8$$
$$= (4\Sigma_3^2 + 30\Sigma_3 - 6)^2 - 44\Sigma_3^2 - 82\Sigma_3 - 32$$
$$> (4\Sigma_3^2 + 30\Sigma_3 - 12)^2 > 0.$$

Thus, our algebraic method has the result that for every three positive integers x_1, x_2, x_3 so that $x_i x_j + 1$ is a square for $1 \le i < j \le 3$ there always exists a fourth positive integer (and usually two distinct fourth integers) x_4 so that $x_i x_4 + 1$; $i = 1, 2, 3$, is a square. Finally, there always exists a fifth rational number, x_5, always a proper fraction, so that $x_i x_5 + 1$; $i =$ 1, 2, 3, 4 is a square.

The question of finding more than four positive integers remains open.

5. Solutions of $x_i x_3' + 1 = y_{i3}'^2$; $i = 1,2$ with $x_3', y_{i3}' \in K = k(x_1, x_2, y_{12})$. The field $K = k(x_1, x_2, y_{12})$ is, of course, the pure transcendental extension $k(x_1, y_{12})$. Sections 4 and 5 show that K contains many solutions x_3', y_{i3}' of equation (1) that are not in $R = k[x_1, x_2, y_{12}]$ and, therefore, are not given in Theorem 1.

For example, we may define a quadruple $x_1, x_2, x_3 = x_3(n)$, $x_4 = x_3(n + 1)$ which satisfies Theorem 3 and then define

$$x_3'(n) = x_5 = \frac{1}{(\sigma_4 - 1)^2}[2\sigma_1 + 4\sigma_3 + 2\sigma_1\sigma_4]$$

as in (10) to get an infinite sequence of triples $x_1, x_2, x_3'(n) \in K$ which satisfy (1). The triple $x_1, x_2, x_3'(n)$ can be augmented, by Theorem 3, to a quadruple $x_1, x_2, x_3(n)$, $x_4'(n)$, where $x_4'(n)$ has the same denominator

$$[\sigma_4(n) - 1]^2 = [x_1 x_2 x_3(n) x_3(n + 1) - 1]^2$$

as $x_3'(n)$. By Theorem 5, this quadruple can be augmented to a quintuple

$$x_1, x_2, x_3'(n), x_4'(n), x_5'(n).$$

Once this process is completed we can start anew, beginning with the triples $x_1, x_2, x_4'(n)$ or $x_1, x_2, x_5'(n)$. Each of the triples can be augmented to quadruples and quintuples, etc. In short, the family of solutions of (1) with x_3, $y_{13}, y_{23} \in K$ appears to be very large, and is quite difficult to characterize completely.

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