DEFINITIONS

The Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) satisfy
\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1
\]
and
\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]
Also, \( a \) and \( b \) designate the roots \((1 + \sqrt{5})/2\) and \((1 - \sqrt{5})/2\), respectively, of
\[
x^2 - x - 1 = 0.
\]

PROBLEMS PROPOSED IN THIS ISSUE

B-412 Proposed by Phil Mana, Albuquerque, NM

Find the least common multiple of the integers in the infinite set
\[
\{2^9 - 2, \ 3^9 - 3, \ 4^9 - 4, \ldots, \ n^9 - n, \ldots\}.
\]

B-413 Proposed by Herta T. Freitag, Roanoke, VA

For every positive integer \( n \), let \( U_n \) consist of the points \( j + k \times 2\pi i/3 \)
in the Argand plane with \( j \in \{0, 1, 2, \ldots, n\} \) and \( k \in \{0, 1, \ldots, j\} \). Let
\( T(n) \) be the number of equilateral triangles whose vertices are subsets of \( U_n \).
For example, \( T(1) = 1, T(2) = 5, \) and \( T(3) = 13 \).
   a. Obtain a formula for \( T(n) \);
   b. Find all \( n \) for which \( T(n) \) is an integral multiple of \( 2n + 1 \).

B-414 Proposed by Herta T. Freitag, Roanoke, VA

Let \( S_n = L_{n+3} + \binom{n}{2} L_{n+2} - \sum_{i=2}^{n} \binom{i}{2} L_i - 11 \). Determine all \( n \) in \( \{2, 3, 4, \ldots\} \) for which \( S_n \) is
(a) prime; (b) odd.

B-415 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA

The circumference of a circle in a fixed plane is partitioned into \( n \) arcs of equal length. In how many ways can one color these arcs if each arc must be red, white, or blue? Colorings which can be rotated into one another should be considered to be the same.
B-416 Proposed by Gene Jakubowski and V. E. Hoggatt, Jr.
San Jose State University, San Jose, CA

Let \( F_n \) be defined for all integers (positive, negative, and zero) by
\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n,
\]
and hence
\[
F_n = F_{n+2} - F_{n+1}.
\]
Prove that every positive integer \( m \) has at least one representation of the form
\[
m = \sum_{j=0}^{n} \alpha_j F_j
\]
with each \( \alpha_j \) in \{0, 1\} and \( \alpha_j = 0 \) when \( j \) is an integral multiple of 3.

B-417 Proposed by R. M. Grassl and P. L. Mana
University of New Mexico, Albuquerque, NM

Here let \([x]\) be the greatest integer in \( x \). Also, let \( f(n) \) be defined by
\[
f(0) = 1 = f(1), \quad f(2) = 2, \quad f(3) = 3, \quad \text{and} \quad f(n) = f(n-4) + [1 + (n/2) + (n^2/12)]
\]
for \( n \in \{4, 5, 6, \ldots\} \). Do there exist rational numbers \( a, b, c, d \) such that
\[
f(n) = [a + bn + cn^2 + dn^3]?
\]

SOLUTIONS

Partitioning Squares Near the Diagonals

B-388 Proposed by Herta T. Freitag, Roanoke, VA

Let \( T_n \) be the triangular number \( n(n+1)/2 \). Show that
\[
T_1 + T_2 + T_3 + \cdots + T_{2n-1} = 1^2 + 2^2 + 3^2 + \cdots + (2n-1)^2
\]
and express these equal sums as a binomial coefficient.

Solution by Phil Mana, Albuquerque, NM

It is readily seen that \( T_1 = 1 = 1^2 \) and \( T_{2k} + T_{2k+1} = (2k+1)^2 \) for \( k = 1, 2, \ldots \). The displayed equation then follows. Next one notes that
\[
T_1 + T_2 + \cdots + T_{2n-1} = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{2n}{2}
\]
\[
= \binom{3}{3} + \left[ \binom{4}{3} - \binom{3}{3} \right] + \cdots + \left[ \binom{2n+1}{3} - \binom{2n}{3} \right]
\]
\[
= \binom{2n+1}{3}.
\]

Also solved by Paul Bracken, Wray G. Brady, Paul S. Bruckman, R. Garfield, Hans Klauser (Switzerland), Peter A. Lindstrom, Graham Lord, Ellen R. Miller, C. B. A. Peck, Bob Prielipp, A. G. Shannon (Australia), Sahib Singh, Paul Smith, Lawrence Somer, Rolf Sonntag (W. Germany), Gregory Wulczyn, and proposer.
Transformed Arithmetic Progression

B-389 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Find the complete solution, with two arbitrary constants, of the difference equation

\[(n^2 + 3n + 3)U_{n+2} - 2(n^2 + n + 1)U_{n+1} + (n^2 - n + 1)U_n = 0.\]

Solution by Paul S. Bruckman, Concord, CA

Let

\[V_n = (n^2 - n + 1)U_n.\]

Then, \[V_{n+1} = (n^2 + n + 1)U_{n+1}, \quad V_{n+2} = (n^2 + 3n + 3)U_{n+2},\]

and so

\[V_{n+2} - 2V_{n+1} + V_n = 0,\]

or

\[\Delta^2 V_n = 0.\]

It follows that \[V_n = an + b,\]

for some constants \(a\) and \(b\). Note that \(V_0 = b = U_0,\) and \(V_1 = a + b = U_1.\)

Hence, \(b = U_0\) and \(a = U_1 - U_0,\)

which implies

\[U_n = \frac{(U_1 - U_0)n + U_0}{n^2 - n + 1}.\]

Also solved by Wray G. Brady, R. Garfield, C. B. A. Peck, Sahib Singh, Paul Smith, and proposer.

Generating Diagonals of Pascal's Triangle

B-390 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA

Find, as a rational function of \(x\), the generating function

\[G_k(x) = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \cdots + \binom{k+n}{k}x^n + \cdots, \quad |x| < 1.\]

I. Solution by Ralph Garfield, College of Insurance; Graham Lord, Université Laval; and Paul Smith, University of Victoria (independently).

\(G_k(x)\) is well known to be \((1 - x)^{k-1}\). (Consider the Taylor series or Newton binomial expansion of this latter function.)

II. Solution by Wray G. Brady, Slippery Rock State College; Robert M. Giuliani, University of California, Santa Cruz, and Herta T. Freitag, Roanoke, VA (independently).

First we show the identity \(G_k(x) = G_{k-1}(x)/(1 - x)\) or

\[G_{k-1}(x) = (1 - x)G_k(x).\]

\[(1 - x)G_k(x) = \binom{k}{k} + \left[\binom{k+1}{k} - \binom{k}{k}\right]x + \left[\binom{k+2}{k} - \binom{k+1}{k}\right]x^2 + \cdots\]

(continued)
Now an induction will show \( G_k(x) = 1/(1-x)^{k+1} \) since \( G_0(x) = 1/(1-x) \).

III. Solution by Paul Bracken (Toronto); Phil Mana; and C. B. A. Peck (independently).

Let \( F_k(x) = (1-x)^{-k-1} \). Then one readily sees that
\[
F_k(x) = (1-x)^{-1} = G_0(x), \quad F_k(0) = 1 = G_k(0),
\]
\[
\frac{dF_k(x)}{dx} = (k+1)F_{k+1}(x), \quad \frac{dG_k(x)}{dx} = (k+1)G_{k+1}(x).
\]
Using integration and induction, one establishes that
\[
G_k(x) = F_k(x) = (1-x)^{-k-1} \quad \text{for} \quad k = 0, 1, 2, \ldots
\]
Also solved by Paul S. Bruckman, A. G. Shannon, Sahib Singh, Gregory Wulczyn, and proposer.

Approximations to Root Five

B-391 Proposed by M. Wachtel, Zurich, Switzerland

Some of the solutions of \( 5x^2 + 1 = y^2 \) in positive integers \( x \) and \( y \) are \((x,y) = (4,9), (72,161), (1292,2889), (23184,51841), \) and \((416020,930249)\). Find a recurrence formula for the \( x_n \) and \( y_n \) of a sequence of solutions \((x_n,y_n)\) and find \( \lim_{n \to \infty} (x_n/y_n) \) in terms of \( a = (1 + \sqrt{5})/2 \).

Solution by Paul S. Bruckman, Concord, California

The Diophantine equation
\[
y^2 - 5x^2 = 1
\]
is a special case of the general Pell equation: \( y^2 - mx^2 = 1 \), where \( m \) is not a square. From the theory of the Pell equation, it is known that (1) possesses infinitely many solutions, and indeed that all of the solutions \((x_n,y_n)\) in positive integers are given by the relation:
\[
y_n + x_n\sqrt{5} = (y_1 + x_1\sqrt{5})^n, \quad n = 1, 2, 3, \ldots,
\]
where \((x_1,y_1)\) is the minimal solution.

We readily find that \((x_1,y_1) = (4,9)\). Let \( A = 9 + 4\sqrt{5} \) and \( B = 9 - 4\sqrt{5} \). Note that \( A = (2 + \sqrt{5})^2 = a^2 \) and \( B = A^{-1} = b^5 \). Since \( y_n - x_n\sqrt{5} = B^n \), it follows that
\[
y_n = (A^n + B^n)/2 = (a^n + b^n)/2,
\]
and
\[
x_n = \frac{1}{2\sqrt{5}}(A^n - B^n) = \frac{a^n - b^n}{2(a-b),}
\]

(3)
\[
(x_n,y_n) = \left(\frac{1}{2}F_{6n^2} + \frac{1}{2}F_{6n^2}, \frac{1}{2}F_{6n^2} \right), \quad n = 1, 2, 3, \ldots
\]

Since \((a - A)(a - B) = a^2 - 18a + 1\), it follows that \( x_n \) and \( y_n \) satisfy the common recursion:
(4) \[ z_{n+2} - 18z_{n+1} + z_n = 0. \]

Moreover, \( L \equiv \lim_{n \to +} (x \cdot z_n) = \lim_{n \to +} \left( \frac{a^{n+1} - B^{n+1}}{A^n - B^n} \right) = A, \) since \( A > 1, \) so \( B < 1, \)

(5) \[ L = a^6. \]

Also solved by Wray G. Brady, C. B. A. Peck, A. G. Shannon, Sahib Singh, Paul Smith, and proposer.

Half-Way Application of \((b^2 - E - 1)^2\

B-392 Proposed by Phil Mana, Albuquerque, NM

Let \( Y_n = (2 + 3n)F_n + (4 + 5n)L_n. \) Find constants \( h \) and \( k \) such that

\[
Y_{n+2} - Y_{n+1} - Y_n = hF_n + kL_n.
\]

Solution by Graham Lord, Université Laval, Québec

\[
Y_{n+2} - Y_{n+1} - Y_n = (2 + 3n + 6)F_{n+2} + (4 + 5n + 10)L_{n+2} - (2 + 3n + 3)F_{n+1} -
\]

\[
-(4 + 5n + 5)L_{n+1} - (2 + 3n)F_n - (4 + 5n)L_n
\]

\[
= 6F_{n+2} - 3F_{n+1} + 10L_{n+2} - 5L_{n+1} = 20F_n + 14L_n.
\]

Thus \( h = 20 \) and \( k = 14. \)


Triangle of Triangular Factorials

B-393 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA

Let \( T_n = \binom{n+1}{2}, P_0 = 1, P_n = T_1T_2 \ldots T_n \) for \( n > 0 \) and \( \left[ \binom{n}{k} \right] = \frac{P_n}{P_k P_{n-k}} \)

for integers \( k \) and \( n \) with \( 0 < k < n. \) Show that

\[
\left[ \binom{n}{k} \right] = \frac{1}{n-k+1} \binom{n}{k} \binom{n+1}{k+1}.
\]

Solution by Paul S. Bruckman, Concord, CA

\[
P_n = \prod_{k=1}^{n} T_k = \prod_{k=1}^{n} k(k+1)/2 = 2^{-n}n!(n+1)! \]

Therefore

\[
\left[ \binom{n}{k} \right] = \frac{P_n}{P_k P_{n-k}} = \frac{n!(n+1)!2^{n-k}}{2^n k!(k+1)!(n-k)!(n+1-k)!}
\]

\[
= \frac{n!}{k!(n-k)!} \cdot \frac{(n+1)!}{(k+1)!(n+1-k)!} = \binom{n}{k} \binom{n+1}{k+1}.
\]


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